The Alberta High School Mathematics Competition

Part II, February 5th , 2020

Problem 1

Let $P(x)$ be a polynomial with integer coefficients. Show that if $P(\frac{1}{3})$ is an even integer then $P(3)$ will also be an even integer.

Solution: If $P(x) = a_n x^n + \cdots + a_1 x + a_0$, and $P(\frac{1}{3}) = M$ where M is even, then

$$a_n \left(\frac{1}{3}\right)^n + a_{n-1}\left(\frac{1}{3}\right)^{n-1} + \cdots + a_1 \frac{1}{3} + a_0 = M$$

or equivalently

$$a_n + a_{n-1} \cdot 3 + \cdots + a_1 \cdot 3^{n-1} + a_0 \cdot 3^n = M \cdot 3^n$$

Thus,

$$\begin{aligned} P(3) - 3^n M &= a_n \cdot 3^n + a_{n-1} \cdot 3^{n-1} + \cdots + a_1 \cdot 3 + a_0 - (a_n + a_{n-1} \cdot 3 + \cdots + a_1 \cdot 3^{n-1} + a_0 \cdot 3^n) \\ &= a_n \cdot (3^n - 1) + a_{n-1} \cdot (3^{n-1} - 3) + \cdots + a_1 \cdot (3 - 3^{n-1}) + a_0 \cdot (1 - 3^n) \end{aligned}$$

Since the $3^n M$ and $3^n - 1, 3^{n-1} - 3, \ldots, 1 - 3^n$ are all even numbers, one obtains that $P(3)$ is an even number.

Alternative Solution: Since $3 \equiv 1 \pmod 2$ then $P(3) \equiv a_0 + a_1 + \cdots + a_n \equiv M3^n \equiv M \pmod 2$ and hence $P(3)$ is even.

Problem 2

Find all functions $f(x) = \frac{47}{ax+b}$, where a and b are integers and $a > 0$, so that $f(4)$ and $f(7)$ are both integers.

Solution: Since $f(4) = \frac{47}{4a+b}$ and $f(7) = \frac{47}{7a+b}$ are both integers, and 47 is prime, $4a + b$ and $7a + b$ must each be equal to one of $\pm 1, \pm 47$. So by subtraction, $3a$ (being positive) must be one of $2, 46, 48$ or 94. The only possibility for integer a is $3a = 48$, which arises from either $7a + b = 47, 4a + b = -1$ or $7a + b = 1, 4a + b = -47$. With $a = 16$, the former gives $b = -65$ and the latter gives $b = -111$. So the possible functions are

$$f(x) = \frac{47}{16x - 65} \qquad \text{and} \qquad f(x) = \frac{47}{16x - 111}.$$

Problem 3

Consider all the subsets of $\{5, 6, 7, \ldots, 15\}$ having at least two elements. How many of these subsets have the property that the sum of the smallest and the largest element in the subset is 20?

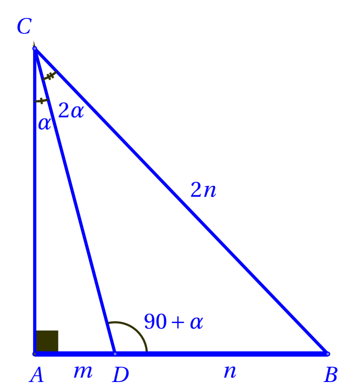
Solution:

Let A_k be a non-empty subset of $\{5, 6, 7, \ldots, 15\}$ having the required property, and with smallest element k, hence $20 - k$ is the largest element of A_k. For the remaining elements in A_k it is possible to choose any (or none) of $k+1, k+2, \ldots, 19-k$. Therefore, the number of such subsets A_k is 2^{19-2k}. On the other hand since $20 - k > k$ and $k \geq 5$, i.e., $5 \leq k \leq 9$, the total number of requested subsets is

$$2^{19-2\cdot5} + 2^{19-2\cdot6} + 2^{19-2\cdot7} + 2^{19-2\cdot8} + 2^{19-2\cdot9} = 2(1 + 4 + 16 + 64 + 256) = 682.$$

Problem 4

$\triangle ABC$ has right angle at A. Point D lies on AB, between A and B, such that $3\angle ACD = \angle ACB$ and $BC = 2BD$. Find the ratio $\dfrac{DB}{DA}$.



Solution:

Let $DA = m, DB = n$, and $\angle ACD = \alpha$. Then $BC = 2n$, and $\angle DCB = 2\alpha$. If the Sine Law is applied in $\triangle DCB$ then

$$\frac{n}{\sin(2\alpha)} = \frac{2n}{\sin(90 + \alpha)} \iff \frac{n}{2\sin(\alpha)\cos(\alpha)} = \frac{2n}{\cos(\alpha)} \iff \sin(\alpha) = \frac{1}{4}$$

and since $CD = \frac{m}{\sin\alpha}$ one obtains $CD = 4m$. Now

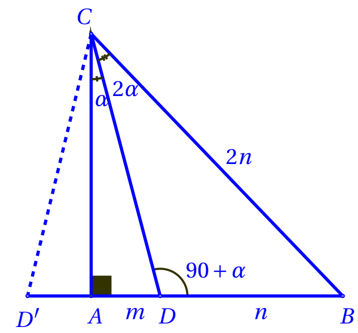
$$AC^2 = (4m)^2 - m^2 = (2n)^2 - (m+n)^2 \Rightarrow 3n^2 - 2mn - 16m^2 = 0 \iff (3n - 8m)(n + 2m) = 0$$

and hence $\frac{DB}{DA} = \frac{n}{m} = \frac{8}{3}$.

Alternative Solution: Take D' the symmetrical of D with respect to AC, then $D'A = m$ and CD is the bisector of $\angle D'CB$. Also, $\triangle D'CD$ is isosceles (CA is the bisector and the altitude) hence $CD' = CD$. In $\triangle D'CB$ by using the Bisector Theorem one obtains

$$\frac{CD'}{CB} = \frac{DD'}{DB} \iff \frac{CD'}{2n} = \frac{2m}{n} \iff CD' = 4m$$

and hence $CD = CD' = 4m$. The solution continues as above.



Problem 5

Let $b_0 < c_0$ be real numbers so that the polynomial $f_0(x) = x^2 + b_0 x + c_0$ has two real roots $b_1 < c_1$ (that is, $f_0(b_1) = f_0(c_1) = 0$) and let $f_1(x) = x^2 + b_1 x + c_1$. If $f_1(x)$ has two real roots $b_2 < c_2$, a new quadratic polynomial $f_2(x) = x^2 + b_2 x + c_2$ is constructed. The process is continued until the quadratic polynomial $f_{n-1}(x) = x^2 + b_{n-1}x + c_{n-1}$, $b_{n-1} < c_{n-1}$ has two real roots $b_n < c_n$, but $f_n(x) = x^2 + b_n x + c_n$, $n \geq 1$ has no real roots.

(a) Show that $n \leq 3$.

(b) Show that $n = 3$ is a possible value.

Solution:

Notice that if x_1, x_2 are the solutions of the quadratic equation $x^2 + Sx + P = 0$ then $x_1 + x_2 = -S$ and $x_1 x_2 = P$; this known fact, which can be immediately justified by using the identity $x^2 + Sx + P = (x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1 x_2$, will systematically be used in the solution of this problem.

(a) Note that one of the inequalities $b_1 \leq 0 \leq c_1, b_1 < c_1 < 0, 0 < b_1 < c_1$ must hold. We prove the following statements:

(A) If $b_1 \leq 0 \leq c_1$ then $f_1(x)$ has no real roots(hence $n = 1$).

Proof of the statement **(A)**:

Since $b_0 = -(b_1 + c_1)$, $c_0 = b_1 c_1$ and $b_0 < c_0$ one obtains $b_1 + c_1 + b_1 c_1 > 0$ and thus $0 \geq b_1 > \frac{-c_1}{1+c_1}$. Hence

$$\Delta_1 = b_1^2 - 4c_1 < \frac{c_1^2}{(c_1+1)^2} - 4c_1 = -c_1 \frac{4c_1^2 + 7c_1 + 4}{(c_1+1)^2} \leq 0,$$

that is $\Delta_1 < 0$, and therefore $f_1(x) = x^2 + b_1 x + c_1$ has no real roots.

(B) If $b_1 < c_1 < 0$ and $f_1(x)$ has real roots $b_2 < c_2$ then $f_2(x) = x^2 + b_2 x + c_2$ has no real roots (hence $n = 2$).

Proof of the statement **(B)**:

Since $b_2 c_2 = c_1 < 0$ we get $b_2 < 0 < c_2$, consequently, following the statement **(A)** and its proof (by replacing the subscript 0 with 1 and 1 with 2) one obtains that $f_2(x) = x^2 + b_2 x + c_2$ has no real roots.

(C) If $0 < b_1 < c_1$ and $f_1(x), f_2(x)$ have real roots $b_2 < c_2$ and respectively $b_3 < c_3$, then $f_3(x) = x^2 + b_3 x + c_3$ has no real roots (hence $n = 3$).

Proof of the statement **(C)**:

From $b_2 + c_2 = -b_1 < 0$ and $b_2 c_2 = c_1 > 0$ we get $b_2 < c_2 < 0$. Now, according to **(B)** (by replacing the subscript 1 with 2 and 2 with 3), one obtains that $f_3(x) = x^2 + b_3 x + c_3$ has no real roots.

Thus $n \leq 3$ in all three cases.

(b) The maximum value of n may be obtained if we can find two real numbers b_1, c_1 so that the conditions in the statement **(C)** are verified, i.e., $0 < b_1 < c_1$ and both polynomials $f_1(x), f_2(x)$ have two real roots $b_2 < c_2$ and respectively $b_3 < c_3$. We will find all these possibilities. Notice that for a selection of b_1, c_1 as above, we have $b_0 = -(b_1 + c_1)$ and $c_0 = b_1 c_1$.

Put $b_1 = a, c_1 = a + h$, such that $a, h > 0$. The quadratic polynomial $f_1(x)$ has distinct real roots if and only if

$$\Delta_1 = b_1^2 - 4c_1 > 0 \iff a^2 - 4a - 4h > 0 \iff (a-2)^2 > 4(h+1) \iff a > 2 + 2\sqrt{1+h},$$

while $f_2(x)$ has real distinct roots if and only if $\Delta_2 = b_2^2 - 4c_2 > 0$. Since $b_2 < c_2$ are the roots of $f_1(x) = x^2 + b_1 x + c_1$, we know that $b_2 = \frac{-b_1 - \sqrt{b_1^2 - 4c_1}}{2} = \frac{-a - \sqrt{\Delta_1}}{2}$ and $c_2 = \frac{-a + \sqrt{\Delta_1}}{2}$ and hence

$$\Delta_2 = b_2^2 - 4c_2 > 0 \iff \left(\frac{-a - \sqrt{\Delta_1}}{2}\right)^2 > 2(-a + \sqrt{\Delta_1}) \iff a^2 + 2(a-4)\sqrt{\Delta_1} + \Delta_1 + 8a > 0.$$

The last inequality in the above sequence of inequalities is always valid if $a > 2 + 2\sqrt{1+h}$. There are infinitely many ways to choose h and a so that $h > 0$ and $a > 2 + 2\sqrt{1+h}$. Therefore, we conclude that there are infinitely many possibilities to choose b_1 and c_1 such that the conditions required in (**C**) are verified, hence the maximum value $n = 3$ is attainable.