ADVANCED CALCULUS LECTURE NOTES FOR MATHEMATICS 217-317

PART II

THIRD EDITION

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CHAPTER FIVE

FURTHER TOPICS IN INTEGRATION

CHANGES OF VARIABLE IN INTEGRALS

Every $n \times n$ matrix A can be expressed as a product of a finite number of matrices of the form

i.e. every linear function from R^n to R^n can be expressed as a composition of linear functions of the form

$$L_1(x_1,\ldots,x_k,\ldots,x_n) = (x_1,\ldots,ax_k,\ldots,x_n)$$

$$L_2(x_1,\ldots,x_k,\ldots,x_n) = (x_1,\ldots,x_k+x_j,\ldots,x_n)$$

$$L_3(x_1,\ldots,x_k,\ldots,x_j,\ldots,x_n) = (x_1,\ldots,x_j,\ldots,x_k,\ldots,x_n) \ .$$

We will not prove this statement. You should prove it yourself as an exercise; it is obviously true for $n = 1$, use induction on n. But first prove it for $n = 2$ to see how the general proof should go. Recall:

Interval in R^n : $I = [\alpha_1,\beta_1] \times \ldots \times [\alpha_n,\beta_n]$

Content of I : $\mu(I) = \prod\limits_{i=1}^{n} (\beta_i-\alpha_i)$

Diameter of I : $\lambda(I) = [\sum\limits_{i=1}^{n} (\beta_i-\alpha_i)^2]^{1/2}$

$$p,q \in I \Rightarrow |p-q| \le \lambda(I) \ .$$

Definition: If $\beta_i-\alpha_i = a$, $i = 1,\ldots,n$ then I is an <u>n-cube</u> of <u>side</u> a and <u>centre</u> $(\frac{\alpha_1+\beta_1}{2}, \ldots, \frac{\alpha_n+\beta_n}{2})$.

<u>Lemma 5.1.1.</u> If I is an interval in R^n and $\phi : R^n \to R^n$ is linear then

$$\mu(\phi(I)) = |J_\phi| \, \mu(I) \ .$$

<u>Proof:</u> ϕ is a composition of a finite number of linear functions of the type L_1, L_2, L_3. The lemma obviously holds for each of these; the type L_1 are the only ones which change the content.

<u>Exercise 5.0:</u> Generalize Lemma 5.1.1 to any set I with content.

Lemmas 5.1.1 - 5.1.4 *and Theorem* 5.1 *are technical in nature. In view of Lemma* 5.1.1 *you will find it easy to accept Theorem* 5.1 *so, on your first reading, you may proceed directly to the important Theorem* 5.2. *Read the statement of Theorem* 5.1 *first.*

<u>Lemma 5.1.2</u>: Suppose

    (i) G open $\subset R^n$.

    (ii) $\phi : G \to R^n$, $\phi \in C^1(G)$.

Then D compact $\subset G$ and $\mu(D) = 0 \Rightarrow \mu(\phi(D)) = 0$.

<u>Proof</u>: If $\epsilon > 0$, $\exists$ a finite collection of n-cubes I_j , $j = 1,\ldots,m$, $\ni$

$$D \subset \bigcup_{j=1}^{m} I_j \subset G \text{ and } \sum_{j=1}^{m} \mu(I_j) < \epsilon \qquad \text{(Why?)}$$

$$\phi \in C^1(G) \Rightarrow \phi \in C^1(\bigcup_{j=1}^{m} I_j)$$

i.e. the partials of ϕ are continuous on the compact set $\bigcup_{j=1}^{m} I_j$. Therefore,

$$\exists M > 0 \ni |D \phi(p)(u)| \leq \frac{M}{n} |u| \ , \ \forall p \in \bigcup_{j=1}^{m} I_j \ , \ \forall u \in R^n \ .$$

Therefore if $p,q \in I_j$ then

$$|\phi(p) - \phi(q)| \leq M|p-q| \qquad \text{(Apply MVTh to each component of } \phi \text{)}$$

$$\leq M \lambda(I_j) \leq \sqrt{n} \, M[\mu(I_j)]^{1/n} \qquad \text{(recall } I_j \text{ is an n-}\underline{\text{cube}}\text{)}$$

$\therefore \quad \phi(I_j) \subset K_j$, an n-cube, and

$$\mu(K_j) \leq [2\sqrt{n}\ M]^n\ \mu(I_j)$$

$$\therefore \quad \phi(D) \subset \bigcup_{j=1}^{m} K_j$$

and

$$\sum_{j=1}^{m} \mu(K_j) \leq [2\sqrt{n}\ M]^n\ \varepsilon\ .$$

$\therefore \quad \phi(D)$ has zero content.

Lemma 5.1.3: Suppose

(i) G open $\subset R^n$

(ii) $\phi : G \to R^n$, $\phi \in C^1(G)$

(iii) $J_\phi(p) \neq 0$, $\forall\ p \in G$.

Then D compact $\subset G$, D has content => $\phi(D)$ is a compact set with content.

Proof: $\phi \in C(D)$, D compact => $\phi(D)$ compact (Theorem 2.12, p. 71). In partic-
ular $\partial D \subset D$ => $\partial\ \phi(D) \subset \phi(D)$. By Theorem 4.16 (p. 201) and Theorem 4.17 (p.
203) ϕ is locally one-to-one on G and maps open sets onto open sets.

$$\therefore \quad \phi(G) \quad \text{and} \quad \phi(G) - \phi(D) \quad \text{are open.}$$

Therefore if $\phi(p) = q \in \partial\ \phi(D)$ each neighbourhood V of q contains a point
$q_1 \in \phi(G) - \phi(D)$. Therefore each neighbourhood U of p contains a point
$p_1 \in G - D$.

$$\therefore \quad p \in \partial D \text{ so } \partial \phi(D) \subset \phi(\partial D) \quad (*)$$

D has content $<=> \mu(\partial D) = 0$

$$\Rightarrow \mu(\phi(\partial D)) = 0 \quad (\text{Lemma } 5.1.2)$$

$$\Rightarrow \mu(\partial \phi(D)) = 0 \quad (\text{by } *) .$$

$$\therefore \quad \phi(D) \text{ has content.}$$

Note: We have used Theorem 3.5 (p. 115) twice i.e. $D \subset R^n$ has content $<=>$ $\mu(\partial D) = 0$, if D is a bounded set.

Lemma 5.1.4: Suppose

    (i) $K = [-r,r] \times \ldots \times [-r,r]$ i.e. K is the n-cube of side $2r$ and centre $(0,\ldots,0)$

    (ii) $K \subset G$ open

    (iii) $\psi : G \to R^n$, $\psi \in C^1(G)$, $J_\psi(p) \neq 0$, $\forall p \in K$.

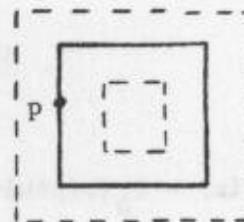Then, if $|\psi(p)-p| < \alpha|p|$, $\forall p \in K$, $0 < \alpha < \dfrac{1}{\sqrt{n}}$,

$$(1-\alpha\sqrt{n})^n < \frac{\mu(\psi(K))}{\mu(K)} < (1+\alpha\sqrt{n})^n .$$

Proof: $\partial \psi(K) \subset \psi(\partial K)$ (cf * above) .

If $p \in \partial K$ then $r \leq |p| \leq \sqrt{n}\, r$

$$\therefore \quad |\psi(p)-p| \leq \alpha|p|$$

$$\leq \alpha\sqrt{n}\, r$$

Therefore $\partial\psi(K)$ lies outside an n-cube of side $2(1-\alpha\sqrt{n})r$, centre $(0,\ldots,0)$, and $\partial\psi(K)$ lies inside an n-cube of side $2(1+\alpha\sqrt{n})r$, centre $(0,\ldots,0)$.

$$\therefore \quad (1-\alpha\sqrt{n})^n \leq \frac{\mu(\psi(K))}{\mu(K)} \leq (1+\alpha\sqrt{n})^n \quad .$$

Theorem 5.1 (The Jacobian Theorem): Suppose

(i) G open $\subset R^n$

(ii) $\phi : G \to R^n$, $\phi \in C^1(G)$, $J_\phi(p) \neq 0 \quad \forall \ p \in G$.

Then, if D compact $\subset G$ and $\varepsilon > 0$, $\exists \ \gamma(\varepsilon) > 0 \ni$ if K is an n-cube of centre $p \in D$ and side length less than 2γ,

$$|J_\phi(p)| \ (1-\varepsilon)^n \leq \frac{\mu(\phi(K))}{\mu(K)} \leq |J_\phi(p)| \ (1+\varepsilon)^n \quad .$$

Proof: Since $J_\phi(p) \neq 0$, $[D\phi(p)]^{-1} = \lambda_p$ and $\lambda_p : R^n \to R^n$ (linear), $\frac{1}{J_\phi(p)} = \det[\lambda_p]$. Since $\phi \in C^1(D)$ and D compact the partials of ϕ are uniformly continuous on D and we have the following:

(a) $\exists \ M > 0 \ni |\lambda_p(u)| \leq M|u|$, $\forall \ p \in D$, $\forall \ u \in R^n$ (Why?)

(b) If $\varepsilon > 0 \ \exists \ \delta(\varepsilon) > 0$ (independent of p) $\ni$ if $p \in D$ and $|u| < \delta$ then

$$|\phi(p+u)-\phi(p)-D\phi(p)(u)| \leq \frac{\varepsilon}{M\sqrt{n}} \ |u| \quad \text{(from M.V. Theorem)}$$

Define $\psi(u) = \lambda_p(\phi(p+u)-\phi(p))$, p fixed.

$$\therefore \quad |\psi(u)-u| \leq \frac{\varepsilon}{\sqrt{n}} |u| \quad \text{(from (a),(b))}$$

$$\therefore \quad (1-\varepsilon)^n \leq \frac{\mu(\psi(K))}{\mu(K)} \leq (1+\varepsilon)^n \quad \text{(from Lemma 5.1.4, } \alpha = \frac{\varepsilon}{\sqrt{n}} \text{)} \quad .$$

But, by Lemma 5.1.1, (and Exercise 5.0) $\mu(\psi(K)) = \det[\lambda_p] \, \mu(\phi(K)) = \frac{\mu(\phi(K))}{J_\phi(p)}$.

$$\therefore \quad |J_\phi(p)|(1-\varepsilon)^n \leq \frac{\mu(\phi(K))}{\mu(K)} \leq J_\phi(p)(1+\varepsilon)^n \quad .$$

<u>Theorem 5.2</u> (Change of Variable Theorem): Suppose

    (i) G open $\subset R^n$

    (ii) $\phi : G \to R^n$, $\phi \in C^1(G)$, $J_\phi(p) \neq 0 \;\; \forall \; p \in G$.

    (iii) D compact, connected $\subset G$, D has content and ϕ is one-to-one

        on D .

Then $\phi(D)$ has content and $f : \phi(D) \to R$, $f \in C(\phi(D)) \Rightarrow$
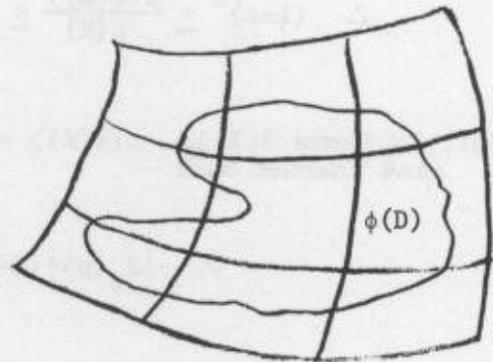
$$\int_{\phi(D)} f = \int_D (f \circ \phi) \, |J_\phi| \quad .$$

<u>Proof:</u> $\phi(D)$ has content by Lemma 5.3. We may assume $f \geq 0$, $J_\phi \geq 0$ (Why?).
$\int_{\phi(D)} f$, $\int_D (f \circ \phi) \, |J_\phi|$ both exist (Theorem 3.4, p. 114); it remains to show
they are equal. For any $\varepsilon > 0$ $(\varepsilon < 1)$ we may choose a partition on D con-
sisting of n-cubes K_j , $j = 1, \ldots, m$, which are sufficiently small so that:

    (a) $\left| \int_D (f \circ \phi) J_\phi - \sum_{j=1}^{m} (f \circ \phi)(q_j) J_\phi(p_j) \mu(K_j) \right| < \varepsilon$ for any $q_j \in K_j$ and

        p_j the centre of K_j .

(b) $J_\phi(p_j)(1-\epsilon)^n \leq \dfrac{\mu(\phi(K_j))}{\mu(K_j)} \leq J_\phi(p_j)(1+\epsilon)^n$.

Now
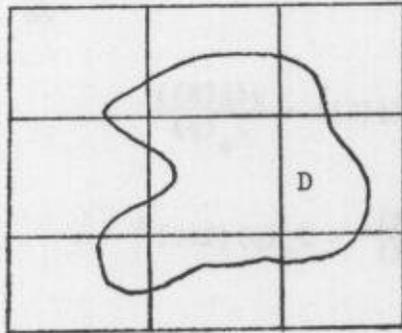
$$\int_{\phi(D)} f = \sum_{j=1}^{m} \int_{\phi(K_j)} f \qquad\qquad (\phi(1-1) \text{ needed here})$$

$$= \sum_{j=1}^{m} f(p_j')\mu(\phi(K_j)) , \; p_j' \in \phi(K_j) \qquad (\text{M.V. Theorem for integrals})$$

$$= \sum_{j=1}^{m} f(\phi(q_j))\mu(\phi(K_j)) , \; q_j \in K_j .$$

$\therefore$ (c) $\displaystyle\sum_{j=1}^{m} f(\phi(q_j)J_\phi(p_j)\mu(K_j)(1-\epsilon)^n \leq \int_{\phi(D)} f \leq \sum_{j=1}^{m} f(\phi(q_j))J_\phi(p_j)\mu(K_j)(1+\epsilon)^n$

from (b). Since ϵ is arbitrary (a),(c) $\Rightarrow \displaystyle\int_D (f\circ\phi) J_\phi = \int_{\phi(D)} f .$ $\square$

Remarks: Strictly speaking the MVTh is not applicable to $\phi(K_j)$ if

$\phi(K_j) \cap \partial\phi(D) \neq \phi$. But this causes no difficulties (Why ?).

You will notice that Theorem 5.2 is much more restrictive than the

corresponding result in R^1 (Corollary 3.8.1, p. 124) which states that

$$\int_{\phi(a)}^{\phi(b)} f = \int_a^b (f \circ \phi) \phi'$$ without the restrictions $\phi' \neq 0$, ϕ is (1-1). These restrictions are necessary in R^n when $n > 1$ because of considerations connected with orientation which we discuss later.

Example 1: Compute the area of the region

$$\{(x,y) : 0 \le y , 0 < r^2 \le x^2 + y^2 \le R^2\} \subset R^2 .$$
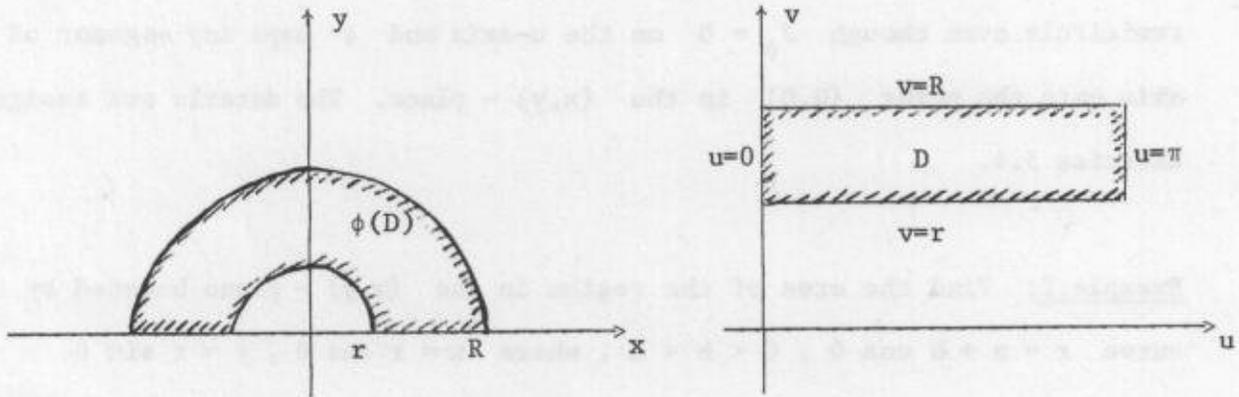
Let $(x,y) = \phi(u,v) = (v \cos u , v \sin u)$

$$J_\phi = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -v \sin u & \cos u \\ v \cos u & \sin u \end{vmatrix} = -v , \quad |J_\phi| = v$$

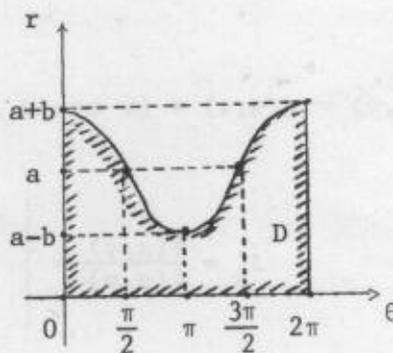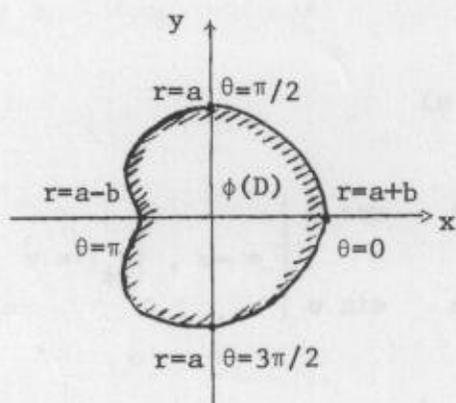$$\mu(\phi(D)) = \int_{\phi(D)} \chi_{\phi(D)} = \int_D (\chi_{\phi(D)} \circ \phi) |J_\phi|$$

$$= \int_D \chi_D |J_\phi| = \int_r^R \int_0^\pi v \, du \, dv$$

$$= \int_r^R \pi v \, dv = \frac{1}{2} \pi \{R^2 - r^2\} .$$

Alternatively you may write this as

$$\int_{\phi(D)} 1 \; dx \; dy = \int_D 1 \; \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \; du \; dv \; .$$

<u>Remark</u>: The conditions "$J_\phi \neq 0$", "ϕ is one-to-one" etc. may fail to hold on a set $S \subset D$ without affecting the result of Theorem 5.2 provided $\mu(S) = \mu(\phi(S)) = 0$. All the integrals involved may be approximated as closely as we please by integrals over regions for which the conditions do hold. For instance, in Example 1, we may take $r = 0$ and get the correct formula for the area of a semicircle even though $J_\phi = 0$ on the u-axis and ϕ maps any segment of that axis onto the point $(0,0)$ in the (x,y) - plane. The details are assigned as Exercise 5.4.

<u>Example 2</u>: Find the area of the region in the (x,y) - plane bounded by the curve $r = a + b \cos \theta$, $0 < b < a$, where $x = r \cos \theta$, $y = r \sin \theta$

$$(x,y) = \phi(r,\theta) = (r \cos \theta, \; r \sin \theta) \; , \quad J_\phi = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$$

$$\int_{\phi(D)} dx\, dy = \int_D \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\, d\theta = \int_0^{2\pi} \int_0^{a+b\cos\theta} r\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (a+b\cos\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (a^2+2ab\cos\theta + b^2\cos^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \{a^2+\tfrac{1}{2}b^2(1+\cos 2\theta)\}\, d\theta$$

$$= \pi(a^2+\tfrac{1}{2}b^2)$$

Here again $J_\phi = 0$ on the θ-axis ($r=0$) and ϕ is not one-to-one on the lines $\theta = 0$, $\theta = 2\pi$, $r = 0$. However $\int_{\phi(D)}$, $\int_D$ both exist and Theorem 5.2 is applicable to regions which differ from $\phi(D)$ and D respectively by regions which have arbitrarily small content.

Example 3: $(x,y) = \phi(u,v) = (u^{2/3}\, v^{1/3}, u^{1/3}\, v^{2/3})$ maps the triangle $D = \{(u,v): 0 \le u, 0 \le v, u + v \le 1\}$ onto the area bounded by the loop of the curve $x^3 + y^3 = xy$
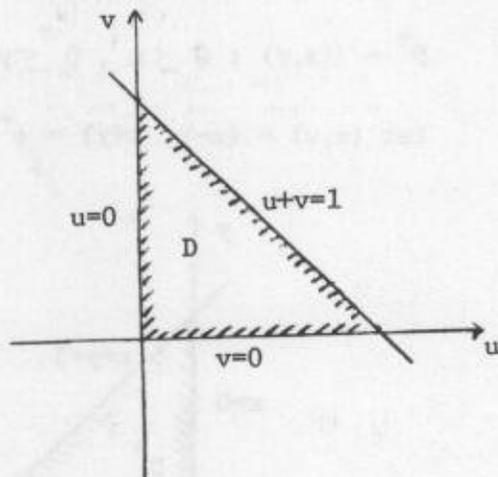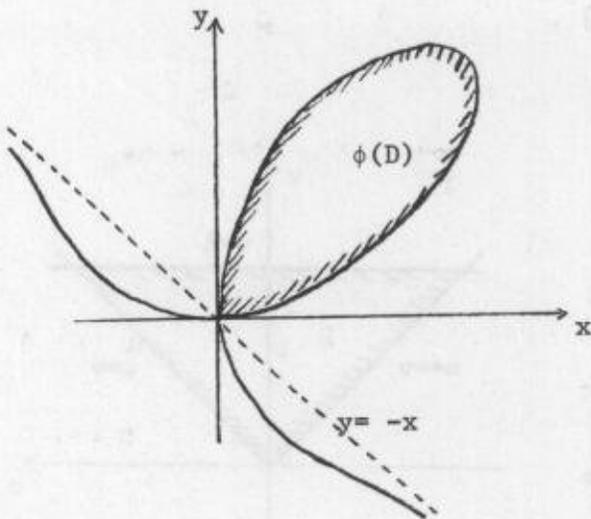
$J_\phi = \frac{1}{3}$; ϕ maps the u and v axes onto $(0,0)$. If f is continuous on $\phi(D)$ then

$$\int_{\phi(D)} f = \int_D (f \circ \phi) |J_\phi|$$

e.g. if $f(x,y) = xy$, $(f \circ \phi)(u,v) = (u^{2/3} v^{1/3})(u^{1/3} v^{2/3}) = uv$

$$\therefore \int_{\phi(D)} f = \frac{1}{3} \int_0^1 \int_0^{1-v} uv \, du \, dv = \frac{1}{6} \int_0^1 (1-v)^2 v \, dv$$

$$= \frac{1}{72} \, .$$

Why was this valid? $\phi \notin C^1$ at $(0,0)$ and ϕ is not $(1-1)$ on the u and v axes.

Examples 1-3 illustrate how the change of variable formula may be used to simplify the region of integration; it may also be used to simplify the integrand. This was its basic role in R^1.

<u>Example 4</u>: Find $I = \int_{D^*} \exp\left(\frac{x-y}{x+y}\right) dx \, dy$ when

$D^* = \{(x,y) : 0 \le x , 0 \le y , x+y \le 1\}$.

Let $(u,v) = (x-y, x+y) = \phi^{-1}(x,y)$
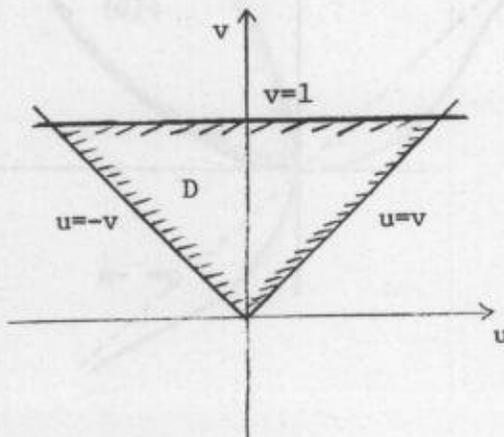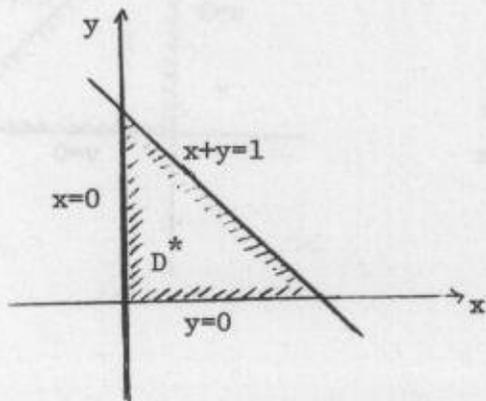
$$D = \phi^{-1}(D^*) \quad D^* = \phi(D)$$

$$(x,y) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right) = \phi(u,v)$$

$$J_\phi = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

Alternatively

$$J_{\phi^{-1}} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \Rightarrow J_\phi = \frac{1}{2}$$

i.e. it was not necessary to find ϕ .

$$\therefore \quad I = \int_D \exp\left(\frac{u}{v}\right) \frac{1}{2} \, du \, dv = \frac{1}{2} \int_0^1 \int_{-v}^v \exp\left(\frac{u}{v}\right) du \, dv$$

$$= \frac{1}{2} \int_0^1 v \, e^{u/v} \Big|_{u=-v}^{u=v} \, dv = \frac{1}{2} \int_0^1 v \left(e - \frac{1}{e}\right) dv$$

$$= \frac{1}{4}\left(e - \frac{1}{e}\right) = \frac{1}{2} \sinh 1 \,.$$

See also the examples in Buck pp 306-311 and the exercises on pp 311-313. The particular exercises 10-12 indicate a proof of Theorem 5.2 based on the Implicit Function Theorem when $n=2$.

Exercises:

5.1: Find $\displaystyle\int_D f_1$, $\displaystyle\int_D f_2$ where $D = \{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$

and $f_1(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$, $f_2(x,y) = x^2 + y^2$

[Solution: $\frac{\pi ab}{2}$, $\frac{\pi ab}{4}(a^2 + b^2)$

5.2:   Let f be a real valued continuous function on R^2 .

Show that

$$\int_0^1 \{\int_0^x f(x,y)\ dy\}\ dx = 2\int_0^{\frac{1}{2}} \{\int_\eta^{1-\eta} f(\xi+\eta\ ,\ \xi-\eta)\ d\xi\}\ d\eta$$

5.3: (i) Show that the ball S = $\{(x,y,z) : x^2 + y^2 + z^2 \le a^2\}$ has content

$M = \frac{4}{3}\pi a^3$ in R^3

[Let x = r sin φ cos θ , y = r sin φ sin θ , z = r cos φ ,

0 ≤ r ≤ a , 0 ≤ θ ≤ 2π , 0 ≤ φ ≤ π]

(ii) Let S be as in (i); show $I_z = \int_S (x^2 + y^2)\ dx\ dy\ dz = \frac{2\ M\ a^2}{5}$

[I_z is the moment of inertia of a uniform spherical mass

distribution (total mass = M) about a diameter.

$\sqrt{\frac{2a^2}{5}}$ is called the radius of gyration about a diameter]

5.4:   (The case of the vanishing Jacobian)

Suppose φ : $R^n \to R^n$ is of the class C^1 on an open set $G \subset R^n$

and that $J_\phi \ne 0$ and φ is (1-1) except on a set K with content

zero. Suppose D ⊂ G is compact and has content and f is a real

valued continuous function on φ(D). Show that φ(D) has content and

$$\int_{\phi(D)} f = \int_D (f\circ\phi)|J_\phi|$$

[Lemma 5.2: If ε > 0 enclose K in the union U of a finite

collection of n cubes such that μ(U) < ε. Apply Theorem 5.2 to

$\overline{D - U}$.

5.5:   Show that $\int_K [(x-y)^2 + 2(x+y) + 1]^{-\frac{1}{2}} \, dx \, dy = 2 \log 2 - \frac{1}{2}$ where

K is the triangle bounded by $x = 0$, $y = 2$ and $x = y$.

[Hint: $x = u(1+v)$, $y = v(1+u)$]

5.6:   Evaluate $\int_{D_1} f$ and $\int_{D_2} f$ where $f(x,y) = \dfrac{1}{(1 + x^2 + y^2)^2}$

where

(i) D_1 is the region bounded by one loop of the lemniscate

$(x^2 + y^2)^2 - (x^2 - y^2) = 0$.

(ii) D_2 is the triangle with vertices $(0,0)$, $(2,0)$, $(1,\sqrt{3})$

[Solution: $\int_{D_1} f = \dfrac{\pi}{4} - \dfrac{1}{2}$, $\int_{D_2} f = \dfrac{\sqrt{3}}{2} \tan^{-1} \dfrac{1}{2}$]

5.7:   Show $\int_K |xyz| \, dx \, dy \, dz = \dfrac{1}{6} a^2 b^2 c^2$ where

$K = \{(x,y,z) : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \le 1\}$

5.8:   Let $\psi(x,y) = Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C$

(i) Show $\int_K \psi = \dfrac{1}{4} \pi ab (Aa^2 + Bb^2 + 4C)$ where

$K = \{(x,y) : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1\}$. Save yourself some work by using

symmetry to show

$$\int_K xy \, dx \, dy = \int_K x \, dx \, dy = \int_K y \, dx \, dy = 0$$

(ii) Show $\int_H \psi = \frac{1}{4} \pi ab \, (Aa^2 + Bb^2 + 4\psi(x_o, y_o))$

where $H = \{(x,y) : \dfrac{(x-x_o)^2}{a^2} + \dfrac{(y-y_o)^2}{b^2} \le 1\}$

(iii) Show $\int_S (2x^2 + y^2 + 3x - 2y + 4) \, dx \, dy = \dfrac{57\pi}{2}$ where S is

the region bounded by the ellipse $x^2 + 4y^2 - 2x + 8y + 1 = 0$.

(iv) If the plane $\ell x + my + nz = p$ intersects the elliptic

paraboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2\,\dfrac{z}{c}$ show that the volume of the solid

bounded by these two surfaces is

$$\frac{\pi ab}{4c^3 n^4} \, (a^2\ell^2 + b^2 m^2 + 2pcn)^2$$

5.9: Find $\int_D \log(x^2 + y^2) \, dx \, dy$ where $D = \{(x,y) : b^2 \le (x^2 + y^2) \le a^2\}$

and show that the limit as $b \to 0$ of this expression is

$2\pi a^2 (\log a - \frac{1}{2})$.

5.10: Show that $\int_D x^3 y^2 \, dx \, dy = 248\frac{2}{5}$ where D is the region bounded by

the lines $y = \pm 3(x-2)$, $y = \pm 3(x-4)$. [Move the origin to the

centre of the region and use symmetry as much as you can.]

5.11: Show that the area of the region in the first quadrant bounded by

$y^3 = a_1 x^2$, $y^3 = a_2 x^2$, $xy^2 = b_1^3$, $xy^2 = b_2^3$ $(a_1 > a_2 > 0$,

$b_1 > b_2 \ge 0)$ is $\frac{7}{5}(b_1^{15/7} - b_2^{15/7})(a_2^{-1/7} - a_1^{-1/7})$.

5.12: Show that the area of the region bounded by the loop of the curve

$x^3 + y^3 = 3axy$ is $\frac{3}{2} a^2$.

5.13: The transformation $x = u^m \phi(v)$, $y = u^n \psi(v)$ maps the rectangle

$[u_1, u_2] \times [v_1, v_2]$ into a region in the (x,y)-plane. The area

of this region is $\left| (u_1^{m+n} - u_2^{m+n}) \int_{v_1}^{v_2} \frac{m\phi\psi' - n\psi\phi'}{m + n} \right|$ if $m \neq -n$ and

$\left| m(\phi_1\psi_1 - \phi_2\psi_2) \log (\frac{u_1}{u_2}) \right|$ if $m = -n$. What conditions must ϕ

and ψ satisfy if this statement is correct? Prove the statement

under your conditions.

5.14: Find $\int_D xyz \, dx \, dy \, dz$ where D is the region in the first octant

bounded by the cylinder $x^2 + y^2 = 16$ and the plane $z = 3$ i.e.

$D = \{(x,y,z) : 0 \leq x , 0 \leq y , 0 \leq z \leq 3 , x^2 + y^2 \leq 16\}$

5.15: Let G be the region in the first quadrant which is bounded by

the curves $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$, $x^2 - y^2 = 4$. Find

$\int_G F$ where $F(x,y) = xy$.

5.16: Show that $\int_D \phi = \frac{1}{3} pq(ap^2 + bq^2) + 2pq \, \phi(x_o, y_o)$ where D is the

region bounded by the four straight lines $\frac{x-x_o}{p} \pm \frac{y-y_o}{q} = \pm 1$

$(p, q > 0)$ and

$$\phi(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c .$$

5.17: (i) If $X = x + y + z + u$, $XY = y + z + u$, $XYZ = z + u$, $XYZU = u$

show $\quad \dfrac{\partial (x,\ y,\ z,\ u)}{\partial (X,\ Y,\ Z,\ U)} = X^3\ Y^2\ Z$

(ii) Show that

$$\int_0^1 \int_0^{1-u} \int_0^{1-z-u} \int_0^{1-y-z-u} (x + y + z + u)^n\ xyzu\ dx\ dy\ dz\ du$$

$$= \int_0^1 X^{n+7}\ dX \int_0^1 Y^5(1-Y)\ dY \int_0^1 Z^3(1-Z)\ dZ \int_0^1 U(1-U)\ dU$$

$$= \frac{1}{(n+8)7!}$$

5.18: Which is larger $\displaystyle\int_0^1 x^x\ dx$ or $\displaystyle\int_0^1 \int_0^1 (xy)^{xy}\ dx\ dy$?

INTEGRATION ON CURVES AND SURFACES

To extend the notion of integration on subsets of R^n as introduced in Chapter III to subsets of k-dimensional objects in R^n it is necessary to introduce a definition of content for these objects. Rather than deal with integration on general k-dimensional objects we will restrict our attention to 1- and 2-dimensional objects, i.e. curves and surfaces. First a manifold segment is too general a concept for which to hope to define content e.g. the graph of $f(x) = \sin \frac{1}{x}$, $x \in (0,1)$ could not be expected to have finite length so we should probably restrict our attention to the maps of fairly well behaved compact subsets of the parameter space. On the other hand the smoothness requirement on manifold segments is somewhat too restrictive in the sense that we should be able to assign a length to objects like those in the picture on the left
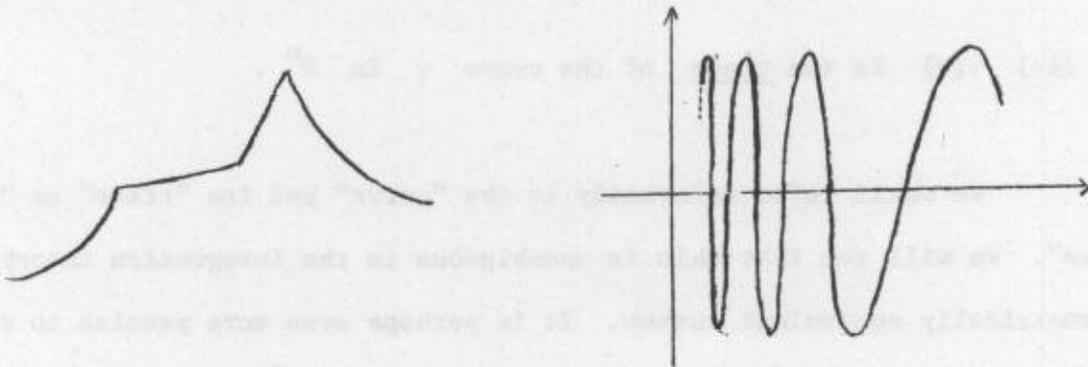
We shall therefore consider objects which behave like "nice" pieces of manifold segments "piecewise".

Curves (1-surfaces):

Definition: Let U be an open connected subset of R and

$$\gamma : U \to R^n, \quad \gamma(t) = (x_1(t), \ldots, x_n(t)), \quad n \geq 1 .$$

(i) If $\gamma \in C(U)$, γ is a <u>curve</u> on U in R^n.

(ii) If $\gamma \in C^1(U)$ and $\mathrm{rk}\, \gamma'(t) = 1$, $\forall\, t \in U$ then γ is a <u>smooth curve</u> on U in R

$$\gamma'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} \qquad \gamma \text{ is smooth} \iff \sum_{i=1}^{n} x_i'(t)^2 > 0 \ \forall\, t \in U$$

(iii) Two smooth curves γ and γ^* on U and U^* are called <u>parametrically</u> <u>equivalent</u> if there is a function $f : U^* \to U$, $f \in C^1(U^*)$, $\ni f'(t) > 0$ and $\gamma^*(t) = \gamma(f(t))$, $\forall\, t \in U^*$.

(iv) $\gamma(U)$ is the <u>trace</u> of the curve γ in R^n.

We shall refer informally to the "curve" and its "trace" as "the curve". We will see that this is unambiguous in the integration theory for parametrically equivalent curves. It is perhaps even more precise to consider curves as equivalence classes of functions $\gamma : R \to R^n$ the equivalence relation being parametric equivalence. However you might consider this idea of a curve too eccentric on your first encounter.

(v) A curve $\gamma : U \to R^n$ is <u>piecewise</u> <u>smooth</u> if

(a) γ is continuous on U

(b) Each compact subinterval of U is the union of a finite number of intervals I such that $\gamma : I^\circ \to R^n$ is a smooth curve and $|\gamma'|$ is Riemann integrable on I .

Example 1:
$$\gamma : \begin{cases} x = x_o + at \\ y = y_o + bt \\ z = z_o + ct \end{cases}, \ t \in R \ , \ \gamma' = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

A line in R^3 . If $a^2 + b^2 + c^2 > 0$ this is a smooth curve.

(a, b, c) are called <u>direction numbers</u> of γ . They are not unique in the sense that $\lambda(a, b, c) = (\lambda a, \lambda b, \lambda c)$ define a parametrically equivalent line if $\lambda > 0$.

$(\ell, m, n) = \dfrac{1}{\sqrt{a^2+b^2+c^2}} (a, b, c)$ are the <u>direction cosines</u> of γ .

$\ell^2 + m^2 + n^2 = 1$

(ℓ,m,n) are the cosines of the angles determined by the line and the directions (1,0,0), (0,1,0), (0,0,1) respectively.
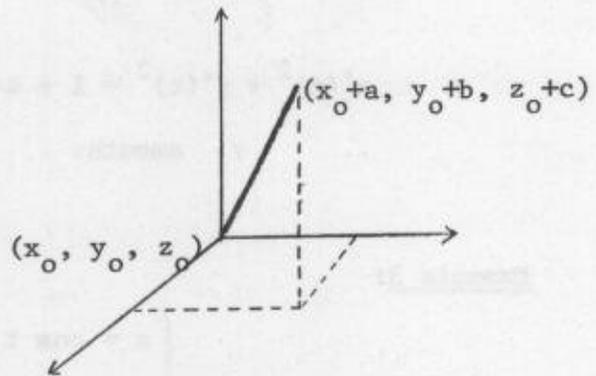
A line is determined by

(i) a <u>point</u> (x_o, y_o, z_o) and direction (a, b, c)

<u>or</u> (ii) <u>two points</u> (x_o, y_o, z_o), (x_1, y_1, z_1) .

i.e. then $(a, b, c) = (x_1 - x_o, y_1 - y_o, z_1 - z_o)$.

The direction (a, b, c) is <u>orthogonal</u> to (α, β, γ) if

$$0 = (a,b,c) \cdot (\alpha,\beta,\gamma) = a\alpha + b\beta + c\gamma$$

The set of points (x, y, z) in R^3 such that $(x-x_o, y-y_o, z-z_o)$ is orthogonal to (a, b, c) is the <u>plane</u> through (x_o, y_o, z_o) with <u>normal</u> (a, b, c). It's equation is

$$0 = (x-x_o, y-y_o, z-z_o) \cdot (a, b, c)$$

i.e. $\quad 0 = a(x-x_o) + b(y-y_o) + c(z-z_o)$
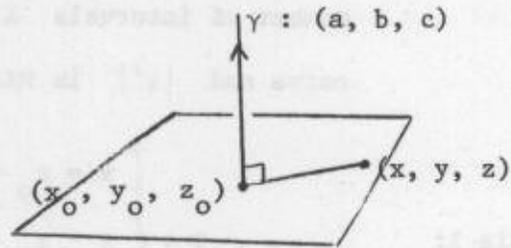
Example 2:

$$\gamma : \begin{cases} x = t \\ \\ y = t^2 \end{cases}, \quad 0 < t < 1, \quad \gamma' = \begin{bmatrix} 1 \\ 2t \end{bmatrix}$$

$$x'(t)^2 + y'(t)^2 = 1 + 4t^2 > 0$$

$\therefore \quad \gamma \quad$ smooth.
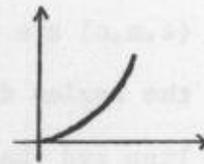
Example 3:

$$\gamma : \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}, \quad t \in R, \quad \gamma' = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$$

γ is a helix

$$x'(t)^2 + y'(t)^2 + z'(t)^2 = 2 > 0$$

$\therefore \quad \gamma \quad$ smooth
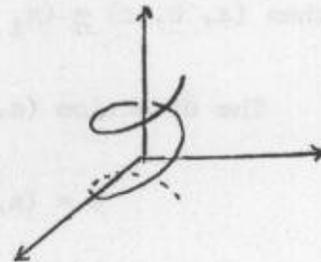
We know that, if $\gamma \in C^1$, $\gamma(t_o) + D\gamma(t_o)(t-t_o)$ is the best affine approximation to $\gamma(t)$ for t near t_o. This motivates the following definition.

<u>Definition</u>: (i) the <u>tangent line</u> to the smooth curve γ at the point t_o is

$$p(t) = \gamma(t_o) + D\gamma(t_o)(t-t_o)$$

e.g. in R^3 : The tangent to the curve

$$\gamma : \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad \text{is} \quad \begin{cases} x = x_o + x'(t_o)(t-t_o) \\ y = y_o + y'(t_o)(t-t_o) \\ z = z_o + z'(t_o)(t-t_o) \end{cases}$$

where $x_o = x(t_o)$, $y_o = y(t_o)$, $z_o = z(t_o)$.

(ii) $\underset{\sim}{v} = \gamma'(t_o)$ is the <u>tangent vector</u> of a smooth curve γ at the point t_o. The <u>direction</u> of a smooth curve γ at the point t_o is

$$\frac{\underset{\sim}{v}}{|\underset{\sim}{v}|} = \frac{\gamma'(t_o)}{|\gamma'(t_o)|}$$

e.g. in R^3 :

$$\frac{\underset{\sim}{v}}{|\underset{\sim}{v}|} = \frac{1}{(x'(t_o)^2 + y'(t_o)^2 + z'(t_o)^2)^{\frac{1}{2}}} (x'(t_o), y'(t_o), z'(t_o))$$

(iii) For a smooth curve in R^3 the <u>normal plane</u> at the point t_o is

$$x'(t_o)(x-x_o) + y'(t_o)(y-y_o) + z'(t_o)(z-z_o) = 0$$

i.e. $[x'(t_o), y'(t_o), z'(t_o)] \begin{bmatrix} x - x_o \\ y - y_o \\ z - z_o \end{bmatrix} = 0$
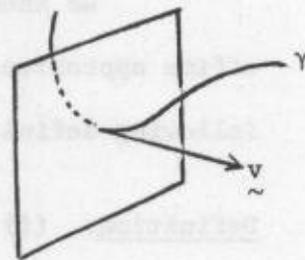
Question: What corresponds to this normal <u>plane</u> for a smooth curve in R^2 ? in R^4 ?

Remark: Two parametrically equivalent curves have the same direction i.e. if $\gamma^*(t) = \gamma(f(t))$, $f' > 0$,

$$\frac{\gamma^{*\prime}(t)}{|\gamma^{*\prime}(t)|} = \frac{\gamma'(f(t))f'(t)}{|\gamma'(f(t))f'(t)|} = \frac{\gamma'(f(t))}{|\gamma'(f(t))|}$$

Definition: (i) If $[a,b] \subset U \subset R$ and γ is a smooth curve on U.

$$\ell(\gamma[a,b]) \overset{\text{def}}{=} \int_a^b |\underset{\sim}{v}| \, dt = \int_a^b |\gamma'(t)| \, dt$$

$$= \int_a^b \sqrt{x_1'(t)^2 + \ldots + x_n'(t)^2} \, dt$$

$\ell(\gamma[a,b])$ is called the <u>length</u> of $\gamma[a,b]$

symbolically: $d\ell = |\underset{\sim}{v}| \, dt = |\gamma'(t)| \, dt$.

    (ii) If γ is a <u>piecewise</u> <u>smooth</u> curve on U then

$$\ell(\gamma[a,b]) = \sum_i \ell(\gamma[a_i, b_i])$$

where $[a_i, b_i]$ are nonoverlapping subintervals of $[a,b]$ such that γ is smooth on (a_i, b_i) .

Remark: If γ^* and γ are parametrically equivalent curves with $f(a^*) = a$, $f(b^*) = b$ then

$$\ell(\gamma^*[a^*, b^*]) = \ell(\gamma[a, b]) \quad \text{since}$$

$$\ell(\gamma^*) = \int_{a^*}^{b^*} |\gamma^{*'}(t)| \, dt = \int_{a^*}^{b^*} |\gamma'(f(t))| f'(t) \, dt$$

$$= \int_a^b |\gamma'(u)| \, du = \ell(\gamma) \, .$$

Example 4: $\gamma : \begin{cases} x = t \\ \\ y = f(t) \end{cases}$ i.e. the curve $y = f(x)$, $\underset{\sim}{v} = (1, f'(t))$

$$\ell(\gamma[a, b]) = \int_a^b \sqrt{1 + f'(t)^2} \, dt$$

Example 5:

$$\gamma : \begin{cases} x = a \cos t \\ \\ y = a \sin t \end{cases}$$

$$\ell(\gamma[0, 2\pi]) = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} \, dt = 2\pi a$$

Note: $\ell(\gamma[0, 4\pi]) = 2 \, \ell(\gamma[0, 2\pi])$ even though these 2 curves have the same trace.

Motivation for the definition of $\ell(\gamma)$:

(a)   The length of the <u>line segment</u>

$$p(t) = p_o + q_o t \quad t \in [t_1, t_2]$$

is $|p(t_2) - p(t_1)| = |q_o(t_2 - t_1)| = |q_o|(t_2 - t_1) = \int_{t_1}^{t_2} |p'(t)| \, dt$

(b)   Consider the curve segment $\gamma[a,b]$ and a partition $\{t_k\}_o^m$ of $[a, b]$. Suppose $t_k \in [t_{k-1}, t_k]$, $k = 1, \ldots, m$; one expects the sum of the lengths of the line segments $p(t) = \gamma(\tau_k) + \gamma'(\tau_k)(t-t_k)$, $t \in [t_{k-1}, t_k]$, $k = 1, \ldots, m$, to approximate what will be the 'length' of $\gamma[a, b]$ as closely as one pleases provided only $\{t_k\}_o^n$ is fine enough.

This sum is

$$\sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1})$$

which is a Riemann sum for

$$\int_a^b |\gamma'(t)| \, dt .$$

Alternatively: the length of $\gamma[a, b]$ is often defined to be

$$\sup \left\{ \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \right\}$$ (The supremum being taken over all partitions $P = \{t_k\}_o^m$ of $[a, b]$). It is not difficult to see that $\ell(\gamma[a,b]) = \int_a^b |\gamma'|$, if $\gamma \in C^1$. This alternative definition is however more general in that it pertains to any curve for which the supremum exists i.e. γ need not be C^1.

<u>Surfaces (2-Surfaces)</u>:

<u>Definition</u>: Let U be an open connected subset of R^2 and

$$\sigma : U \to R^n, \quad \sigma(u,v) = (x_1(u,v), \ldots, x_n(u,v)), \quad n \geq 2 .$$

   (i) If $\sigma \in C(U)$, σ is a <u>surface</u> (2-surface) on U in R^n

   (ii) If $\sigma \in C^1(U)$ and $rk \, \sigma'(u,v) = 2$, $\forall (u,v) \in U$

Then σ is a smooth surface on U in R^n

$$\sigma' = \begin{bmatrix} \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_1}{\partial v} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \dfrac{\partial x_n}{\partial u} & \dfrac{\partial x_n}{\partial v} \end{bmatrix} \qquad \begin{array}{c} \sigma \text{ is smooth} <=> \\[2mm] \displaystyle\sum_{i,j=1}^{n} \left[\dfrac{\partial(x_i, x_j)}{\partial(u, v)}\right]^2 > 0 \ . \end{array}$$

(iii) Two smooth surfaces σ and σ^* on U and U^* respectively are called parametrically equivalent if there is a function $f : U^* \to U$, $f \in C^1(U^*) \ni J_{f(p)} > 0$, f is (1-1) and $\sigma^*(p) = \sigma(f(p))$, $\forall p \in U^*$.

(iv) $\sigma(U)$ is the trace of the surface σ in R^n .

Again we will not worry excessively about distinguishing between a surface and its trace.

(v) A surface $\sigma : U \to R^n$ is piecewise smooth if

(a) σ is continuous on U

(b) Each compact Jordan measurable (has content) subset of U is the union of a finite number of Jordan measurable subsets D such that $D = \overline{D^\circ}$ and $\sigma : D^\circ \to R^n$ is a smooth surface and

$$\left[\sum_{i,j=1}^{n} \left(\frac{\partial(x_i, x_j)}{\partial(u, v)}\right)^2\right]^{\frac{1}{2}} \qquad \text{is Riemann integrable on } D \ .$$

Example 1:

$$\sigma : \begin{cases} x = u + v \\ y = u - v \\ z = u \end{cases}, \ (u,v) \in R^2 \ , \ \sigma' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$(*) \quad x + y - 2z = 0$, a plane in R^3 .

$$\text{rk } \sigma' = 2 \ , \quad \therefore \quad \text{smooth} .$$

We can see from $(*)$ that direction numbers of the normal are $(1, 1, -2)$.

Notice also that these direction numbers are in fact

$$\left(\frac{\partial (y,z)}{\partial (u,v)} , \frac{\partial (z,x)}{\partial (u,v)} , \frac{\partial (x,y)}{\partial (u,v)} \right) .$$

Example 2:

$$\sigma : \begin{cases} x = \cos \theta \sin \phi \\ y = \sin \theta \sin \phi, (\theta,\phi) \in R^2 \\ z = \cos \phi \end{cases}$$

$$x^2 + y^2 + z^2 = 1 \quad \text{a sphere in } R^3$$

$$\sigma' = \begin{bmatrix} -\sin \theta \ \sin \phi & \cos \theta \ \cos \phi \\ \cos \theta \ \sin \phi & \sin \theta \ \cos \phi \\ 0 & -\sin \phi \end{bmatrix} , \text{rk } \sigma' = 2, (\phi \neq m\pi).$$

$$\therefore \quad \text{smooth}$$

Example 3:

$$\sigma : \begin{cases} x = x_o + a_1 u + b_1 v \\ y = y_o + a_2 u + b_2 v \\ z = z_o + a_3 u + b_3 v \end{cases}$$

$$\text{i.e.} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$(x, y, z) = (x_o, y_o, z_o) + u(a_1, a_2, a_3) + v(b_1, b_2, b_3)$ a plane through

(x_o, y_o, z_o) if $\underset{\sim}{a} = (a_1, a_2, a_3)$, $\underset{\sim}{b} = (b_1, b_2, b_3)$ are linearly independent.

$$\frac{\partial (y,z)}{\partial (u,v)} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = A \quad ; \quad \frac{\partial (z,x)}{\partial (u,v)} = \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} = B$$

$$\frac{\partial (x\ y)}{\partial (u,v)} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = C$$

σ is a smooth surface $\iff A^2 + B^2 + C^2 > 0 \iff \underset{\sim}{a}, \underset{\sim}{b}$ linearly independent.

To identify the plane in the form given on page 264 consider

$$A(x - x_o) + B(y - y_o) + C(z - z_o)$$

$$= A(a_1 u + b_1 v) + B(a_2 u + b_2 v) + C(a_3 u + b_3 v)$$

$$= u \begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} + v \begin{vmatrix} b_1 & a_1 & b_1 \\ b_2 & a_2 & b_2 \\ b_3 & a_3 & b_3 \end{vmatrix} = 0$$

This is the plane through (x_o, y_o, z_o) with normal

$$\underset{\sim}{n} = (A, B, C) = \left(\frac{\partial (y,z)}{\partial (u,v)}, \frac{\partial (z,x)}{\partial (u,v)}, \frac{\partial (x,y)}{\partial (u,v)} \right)$$

Notice that the normal is given by

$$\underset{\sim}{n} = \underset{\sim}{a} \times \underset{\sim}{b} \overset{def}{=} \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underset{\sim}{e}_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + \underset{\sim}{e}_2 \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} + \underset{\sim}{e}_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

The best affine approximation to any surface $\sigma(u,v)$ for (u,v) near (u_o, v_o) is $\sigma(u_o, v_o) + D\,\sigma(u_o, v_o)(u - u_o, v - v_o)$; hence the following definition.

Definition:

(i)   The tangent plane to the smooth surface σ at the point (u_o, v_o) is

$$p(u,v) = \sigma(u_o, v_o) + D\,\sigma(u_o, v_o)(u - u_o, v - v_o) \ .$$

For example, in R^3 : The tangent to the surface

$$\sigma : \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \quad \text{is} \quad \begin{cases} x = x_o + \left.\dfrac{\partial x}{\partial u}\right|_{P_o}(u - u_o) + \left.\dfrac{\partial x}{\partial v}\right|_{P_o}(v - v_o) \\[2ex] y = y_o + \left.\dfrac{\partial y}{\partial u}\right|_{P_o}(u - u_o) + \left.\dfrac{\partial y}{\partial v}\right|_{P_o}(v - v_o) \\[2ex] z = z_o + \left.\dfrac{\partial z}{\partial u}\right|_{P_o}(u - u_o) + \left.\dfrac{\partial z}{\partial v}\right|_{P_o}(v - v_o) \end{cases}$$

where $p_o = (u_o, v_o)$, $x_o = x(u_o, v_o)$, $y_o = y(u_o, v_o)$, $z_o = z(u_o, v_o)$.

From Example 3 (p. 270) this is the plane

$$\left.\frac{\partial (u,z)}{\partial (u,v)}\right|_{P_o}(x - x_o) + \left.\frac{\partial (z,x)}{\partial (u,v)}\right|_{P_o}(y - y_o) + \left.\frac{\partial (x,y)}{\partial (u,v)}\right|_{P_o}(z - z_o) = 0$$

(ii)   In R^3 the normal line to the surface σ at p_o has direction numbers

$$\underset{\sim}{n} = \left.\left(\frac{\partial (y,z)}{\partial (u,v)} \ , \ \frac{\partial (z,x)}{\partial (u,v)} \ , \ \frac{\partial (x,y)}{\partial (u,v)} \right)\right|_{P_o}$$

The unit normal direction is $\underset{\sim}{n}_1 = \dfrac{\underset{\sim}{n}}{|\underset{\sim}{n}|}$

Exercises:

5.19:   Show that two parametrically equivalent surfaces in R^3 have the
same unit normal $\underset{\sim}{n}_1$.

5.20:   Let $\sigma : R^2 \rightarrow R^n$ $(x_i = x_i(u,v) \quad i = 1, \ldots, n)$ be a smooth surface.
Show that the vectors

$$\underset{\sim}{u} = \left(\frac{\partial x_1}{\partial u} , \ldots , \frac{\partial x_n}{\partial u} \right)_{P_o} , \quad \underset{\sim}{v} = \left(\frac{\partial x_1}{\partial v} , \ldots , \frac{\partial x_n}{\partial v} \right)_{P_o}$$

are tangent vectors to certain smooth curves in σ . Check that
the condition $\operatorname{rk} \sigma'(p_o) = 2$ simply requires that $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be
linearly independent. Check that in the case $n=3$ $\underset{\sim}{n} = \underset{\sim}{u} \times \underset{\sim}{v}$.

Question: In R^4 what corresponds to the normal line discussed above for
a surface in R^3? If you can't figure this out ask.

We discuss the concept of surface area only for 2-surfaces in R^3 .
It is hoped that the treatment of the problem of surface area in R^n should
be clear from this. It should further indicate the procedure to be adopted
in dealing with content of k-surfaces in R^n .

Definition:

(i)   If D is a compact Jordan measurable (has content) subset of $U \subset R^2$
and σ is a smooth surface on U in R^3

$$\sigma : \begin{cases} x = x(u,v) \\ y = y(u,v), \qquad (u,v) \in U \\ z = z(u,v) \end{cases}$$

then

$$a(\sigma(D)) \overset{\text{def}}{=} \int_D |\underset{\sim}{n}| \, du \, dv$$

$$= \int_D \sqrt{\frac{\partial(y,z)}{\partial(u,v)}^2 + \frac{\partial(z,x)}{\partial(u,v)}^2 + \frac{\partial(x,y)}{\partial(u,v)}^2} \, du \, dv$$

$a(\sigma(D))$ is called the <u>area</u> of $\sigma(D)$. Symbolically $da = |\underset{\sim}{n}| \, du \, dv$.

(ii) If σ is a <u>piecewise</u> <u>smooth</u> surface on U then

$$a(\sigma(D)) = \sum_i a(\sigma(D_i))$$

where D_i are nonoverlapping Jordan measurable subsets of D such that σ is smooth on D_i° .

<u>Remark</u>: Two parametrically equivalent surfaces have the same area (cf. p. 279) .

<u>Example 4</u>: (Surface area for $z = f(x,y)$, $(x,y) \in D$

$$\sigma : \begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases} \quad \text{i.e.} \quad z = f(x,y) \ , \ f \in C^1 .$$

$$\sigma' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ f_u & f_v \end{bmatrix} \qquad \text{rk } \sigma' = 2 \ \therefore \ \text{smooth}$$

$$a(\sigma(D)) = \int_D \sqrt{f_u^2 + f_v^2 + 1} \, du \, dv$$

$$= \int_D \sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy$$

Example 5: (Surface area of a sphere of radius a)

$$\sigma : \begin{cases} x = a \sin \phi \cos \theta \\ y = a \sin \phi \sin \theta \\ z = a \cos \phi \end{cases} \quad D : \begin{cases} 0 \le \theta \le 2\pi \\ \\ 0 \le \phi \le \pi \end{cases}$$

$$x^2 + y^2 + z^2 = a^2$$

$$\sigma' = \begin{bmatrix} -a \sin \phi \sin \theta & a \cos \phi \cos \theta \\ a \sin \phi \cos \theta & a \cos \phi \sin \theta \\ 0 & -a \sin \phi \end{bmatrix}$$

$$a(\sigma(D)) = \int_D |\underline{n}|$$
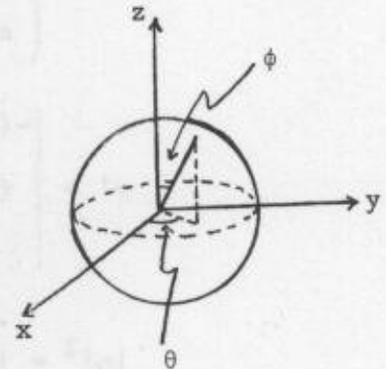
$$= \int_0^\pi \int_0^{2\pi} \sqrt{a^4 \sin^2\phi \cos^2\phi + a^4 \sin^4\phi} \; d\theta \; d\phi$$

$$= \int_0^\pi \int_0^{2\pi} a^2 \sin \phi \; d\theta \; d\phi$$

$$= -2\pi a^2 \cos \phi \Big|_0^\pi = 4\pi a^2 .$$

Example 6: (Surface area of a torus)
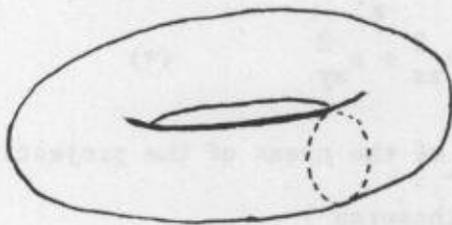
$$\sigma : \begin{cases} x = (R - a \cos \phi) \cos \theta \\ y = (R - a \cos \phi) \sin \theta \ , \\ z = a \sin \phi \end{cases} D : \begin{cases} 0 \leq \theta \leq 2\pi \\ \\ 0 \leq \phi \leq 2\pi \end{cases} , \quad R > a > 0 \ .$$

$$\sigma' = \begin{vmatrix} -(R - a \cos \phi) \sin \theta & a \sin \phi \cos \theta \\ (R - a \cos \phi) \cos \theta & a \sin \phi \sin \theta \\ 0 & a \cos \phi \end{vmatrix}$$

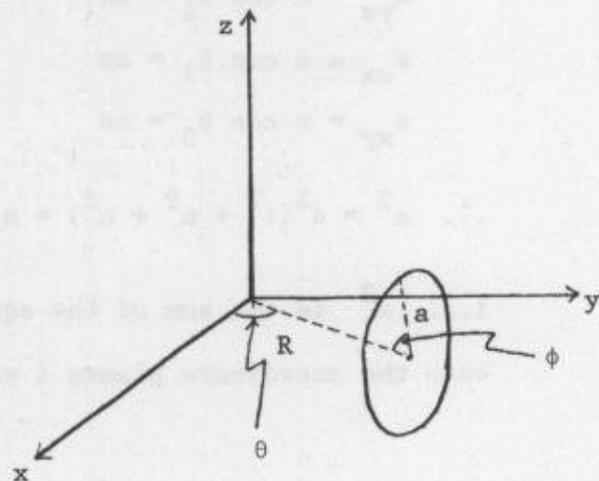$$\begin{aligned} |\underline{n}|^2 = \ & \{-(R - a \cos \phi) a \sin \phi (\sin^2\theta + \cos^2\theta)\}^2 \\ & + \{(R - a \cos \phi) a \cos \phi \cos \theta\}^2 \\ & + \{(R - a \cos \phi) a \cos \phi \sin \theta\}^2 = a^2(R - a \cos \phi)^2 \end{aligned}$$

$$a(\sigma(D)) = \int_D |n| = \int_0^{2\pi} \{\int_0^{2\pi} a(R - a \cos \phi) \, d\theta\} \, d\phi$$

$$= 2\pi a \int_0^{2\pi} (R - a \cos \phi) \, d\phi$$

$$= (2\pi)^2 \, a \, R \ .$$

Motivation for the definition of $a(\sigma)$:

(a) The area of a plane segment

$$a_{yz} = a \cos \theta_1 = a\ell$$

$$a_{zx} = a \cos \theta_2 = am$$

$$a_{xy} = a \cos \theta_3 = an$$

$$\therefore \quad a^2 = a^2(\ell^2 + m^2 + n^2) = a_{yz}^2 + a_{zx}^2 + a_{xy}^2 \qquad (*)$$

i.e. a^2 is the sum of the squares of the areas of the projections

onto the coordinate planes (yea Pythagoras) .

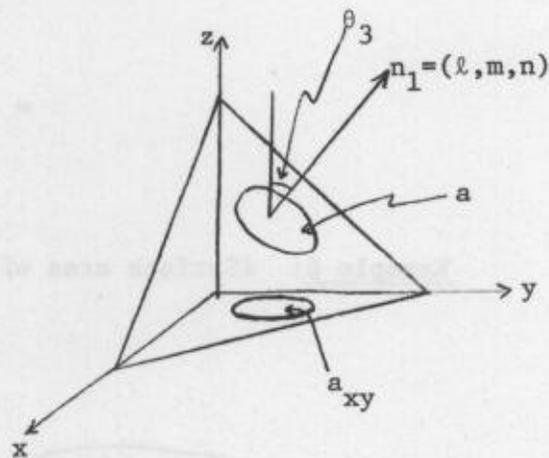Consider an affine function $\sigma : R^2 \rightarrow R^3$ and the projections σ_{yz}, σ_{zx}, σ_{xy} onto the coordinate planes in R^3:

$$\sigma : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\sigma_{yz} : \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y_o \\ z_o \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\sigma_{zx} : \begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} z_o \\ x_o \end{bmatrix} + \begin{bmatrix} a_3 & b_3 \\ a_1 & b_1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\sigma_{xy} : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

If D is a subset of R^2 with content then, from Lemma 5.1.1 (p. 244)

$$a(\sigma_{yz}(D)) = \left\| \begin{matrix} a_2 & b_2 \\ a_3 & b_3 \end{matrix} \right\| a(D) = \left| \frac{\partial (y,z)}{\partial (u,v)} \right| a(D) \text{ , etc.}$$

so, by the remark (*) above,

$$a(\sigma(D)) = \sqrt{ a(\sigma_{yz}(D))^2 + a(\sigma_{zx}(D))^2 + a(\sigma_{xy}(D))^2 }$$

$$= \sqrt{ \frac{\partial (y,z)^2}{\partial (u,v)} + \frac{\partial (z,x)^2}{\partial (u,v)} + \frac{\partial (x,y)^2}{\partial (u,v)} } \; a(D)$$

$$= |\underset{\sim}{n}| \; a(D) = \int_D |\underset{\sim}{n}| \; du \, dv \quad \text{(n \quad constant here)}$$

(b) The area of a smooth surface.

If $\sigma : D \to R^3$ $(D \subset R^2)$ is a smooth surface.

If D is partitioned
into subsets D_i the
sum of the areas of
tangent plane segments
$|\underset{\sim}{n}(p_i)| a(D_i)$, $p_i \in D_i$,
should be expected to
approximate the "area" of $\sigma(D)$ closely provided the partition is
fine enough. But $\underset{i}{\Sigma} |\underset{\sim}{n}(p_i)| a(D_i)$ is a Riemann sum for $\int_D |\underset{\sim}{n}| \, du \, dv$.
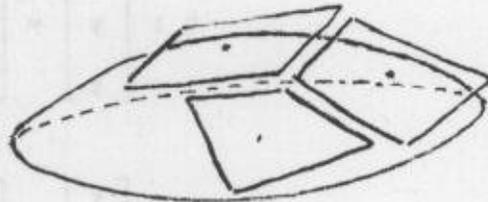
Hence it is reasonable to take $a(\sigma(D)) = \int_D |\underset{\sim}{n}| \, du \, dv$.

Now that the notion of content or measure has been introduced for
curves and surfaces it is a simple matter to extend the idea of integration
to such objects. For example in R^3 : If $\gamma : R \to R^3$ is a smooth curve
and I is a closed interval in R then a partition of $\gamma(I)$ is induced by a
partition of I into subintervals $\{I_i\}$ so that if f is a real valued
function on $\gamma(I)$ we may define Riemann sums

$$\underset{i}{\sum} f(p_i) \ell(\gamma(I_i)) \quad , \quad p_i \in \gamma(I_i)$$

$$= \underset{i}{\sum} (f \circ \gamma)(t_i) \ell(\gamma(I_i)) \quad , \quad t_i \in I_i$$

with resulting integral

$$\int_{\gamma(I)} f = \int_{\gamma(I)} f \, d\ell \overset{\text{def}}{=} \int_I (f \circ \gamma) |\underset{\sim}{v}| \, dt$$

$$= \int_I f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt$$

Similarly if $\sigma : R^2 \to R^3$ is a smooth surface and D a Jordan measurable subset of R^2 then a partition of $\sigma(D)$ may be induced by partitions of D into subsets $\{D_i\}$ so that if f is a real valued function on $\sigma(D)$ we may define Riemann sums

$$\sum_i f(p_i) a(\sigma(D_i)) \quad , \quad p_i \in \sigma(D_i)$$

$$= \sum_i (f \circ \sigma)(u_i, v_i) a(\sigma(D_i)) \quad , \quad (u_i, v_i) \in D_i$$

with resulting integral

$$\int_{\sigma(D)} f = \int_{\sigma(D)} f da \overset{\text{def}}{=} \int_D f \circ \sigma \, |\underset{\sim}{n}| \, du \, dv$$

$$= \int_D f(x,y,z) \sqrt{\frac{\partial (y,z)^2}{\partial (u,v)} + \frac{\partial (z,x)^2}{\partial (u,v)} + \frac{\partial (x,y)^2}{\partial (u,v)}} \, du \, dv \; , \; x = x(u,v) \quad \text{etc.}$$

Remark: If γ and γ^* (σ and σ^*) are parametrically equivalent curves (surfaces) then

$$\int_{\gamma(I)} f = \int_{\gamma^*(I^*)} f \left(\int_{\sigma(D)} f = \int_{\sigma^*(D^*)} f \right)$$

Proof (Optional):

$$\underline{\int_{\gamma(I)} f = \int_{\gamma^*(I^*)} f :} \qquad \exists \, \phi \in C^1 : I^* \to I \ni$$

$$\phi'(t) > 0 \quad \text{and} \quad \gamma^*(t) = \gamma(\phi(t)) \; , \; \forall \, t \in I^* \; .$$

$$\therefore \int_{\gamma^*(I^*)} f \overset{def}{=} \int_{I^*} f(\gamma^*(t))|\gamma^{*\prime}(t)|dt$$

$$= \int_{I^*} f(\gamma(\phi(t)))|\gamma'(\phi(t))|\phi'(t)\, dt$$

$$= \int_I f(\gamma(u))|\gamma'(u)|du \overset{def}{=} \int_{\gamma(I)} f$$

$$\int_{\sigma(D)} f = \int_{\sigma^*(D^*)} f : \qquad \exists\, \phi \in C^1 : D^* \to D \ni$$

$$J_\phi(p) > 0, \ \phi \text{ is } (1\text{-}1) \text{ and } \sigma^*(p) = \sigma(\phi(p)), \ \forall\, p \in D^*,$$

i.e. $\phi(r,s) = (u(r,s),\, v(r,s))$ and $\dfrac{\partial(u,v)}{\partial(r,s)} > 0, \ \forall(r,s) \in D^*,$

$$\sigma^*(r,s) = (x(r,s),\, y(r,s),\, z(r,s)) = (X(u,v),\, Y(u,v),\, Z(u,v))$$

$$= \sigma(u,v) \qquad u = u(r,s), \qquad v = v(r,s)$$

$$\int_{\sigma^*(D^*)} f \overset{def}{=} \int_{D^*} f(x,y,z)\sqrt{\frac{\partial(y,z)^2}{\partial(r,s)} + \frac{\partial(z,x)^2}{\partial(r,s)} + \frac{\partial(x,y)^2}{\partial(r,s)}}\, dr\, ds$$

$$= \int_{D^*} f(X,Y,Z)\sqrt{\frac{\partial(Y,Z)^2}{\partial(u,v)} + \frac{\partial(Z,X)^2}{\partial(u,v)} + \frac{\partial(X,Y)^2}{\partial(u,v)}}\ \frac{\partial(u,v)}{\partial(r,s)}\, dr\, ds \quad \text{(chain rule)}$$

$$= \int_D f(X,Y,Z)\sqrt{\frac{\partial(Y,Z)^2}{\partial(u,v)} + \frac{\partial(Z,X)^2}{\partial(u,v)} + \frac{\partial(X,Y)^2}{\partial(u,v)}}\, du\, dv \quad \text{(Theorem 5.2)}$$

$$\overset{def}{=} \int_{\sigma(D)} f\, .$$

For a piecewise smooth curve γ (surface σ)

$$\int_{\gamma(I)} f = \sum_i \int_{\gamma(I_i)} f \quad \left(\int_{\sigma(D)} f = \sum_i \int_{\sigma(D_i)} f \right)$$

where $\gamma(I_i)\ (\sigma(D_i))$ are the portions of $\gamma(I)\ (\sigma(D))$ which are smooth.

You should think briefly about how $\int_{\sigma(D)} f$ may be defined if σ is a smooth k-surface in R^n.

Applications: If S is any set for which the idea of a partition into subsets has been defined and for which measure (or content) μ has been defined for these subsets then we define the average (mean) of a function f on S to be

$$\bar{f} = \frac{\int_S f d\mu}{\int_S d\mu}$$

whenever this exists.

Example 1: Three points are chosen at random on a line segment of length a. Find the average distance of the intermediate point from the midpoint of the segment

$$|\leftarrow \qquad\qquad a \qquad\qquad \rightarrow|$$

$$|\text{---}+\text{---}+\text{---}+\text{---}|$$

$$|\leftarrow x \rightarrow|\leftarrow y \rightarrow|\leftarrow z \rightarrow|$$
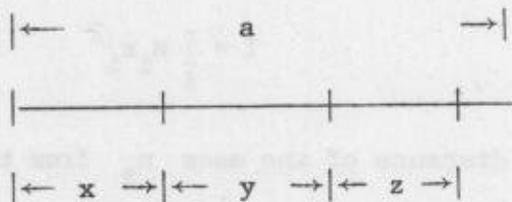
We must find the average value c of $\left|\frac{a}{2} - x - y\right|$ in the set

$$D = \{(x,y,z) : x,y,z \geq 0 , 0 \leq x + y + z \leq a\} \text{ (tetrahedron) i.e.}$$

$$c = \int_D \left|\frac{a}{2} - x - y\right| dx\, dy\, dz \bigg/ \int_D dx\, dy\, dz$$

$$\int_D dx\, dy\, dz = \int_0^a \int_0^{a-x} \int_0^{a-x-y} dx\, dy\, dz = \frac{a^3}{6}$$

$$\int_D \left|\frac{a}{2} - x - y\right| \, dx \, dy \, dz = \int_0^a \int_0^{a-x} \int_0^{a-x-y} \left|\frac{a}{2} - x - y\right| dz \, dy \, dx$$

$$= \int_0^a \int_0^{a-x} \left|\frac{a}{2} - x - y\right| (a - x - y) dy \, dx$$

$$= \int_0^a \int_0^{\frac{a}{2}-x} \left(\frac{a}{2} - x - y\right)(a - x - y) dy \, dx + \int_0^a \int_{\frac{a}{2}-x}^{a-x} \left(x + y - \frac{a}{2}\right)(a - x - y) dy \, dx$$

$$= \frac{a^4}{32}$$

$$\therefore \quad c = \frac{a^4}{32} \bigg/ \frac{a^3}{6} = \frac{3a}{16}$$

For a finite distribution of point masses m_i at the points (x_i, y_i, z_i) the coordinates of the centre of mass (centroid) are

$$\bar{x} = \sum_i m_i x_i \bigg/ \sum_i m_i , \quad \bar{y} = \sum_i m_i y_i \bigg/ \sum_i m_i , \quad \bar{z} = \sum_i m_i z_i \bigg/ \sum_i m_i$$

The moment of inertia of such a distribution about an axis is

$$I = \sum_i m_i r_i^2$$

where r_i is the distance of the mass m_i from the axis. The radius of gyration is

$$\sigma = \sqrt{\sum_i m_i r_i^2 \bigg/ \sum_i m_i} = \sqrt{I/m}$$

The significance of the centroid is that a mass distribution is equivalent statically (i.e. with regard to first moments) to a point mass of the same magnitude located at its centroid. The moment of inertia and radius of gyration are important in dynamics. A point mass m rotating with angular velocity ω about an axis a distance r away has kinetic energy

$$\frac{1}{2} mv^2 = \frac{1}{2} m(r\omega)^2 = \frac{1}{2} I\omega^2 \qquad (v = r\omega)$$

Thus a finite number of particles rotating about axis with angular velocity ω has kinetic energy $\frac{1}{2} \sum m_i (r_i \omega)^2 = \frac{1}{2} I\omega^2$ and has the same energy as a single point mass $m = \sum_i m_i$ situated a distance $\sigma (=$ radius of gyration) from the axis and rotating about it with angular velocity ω .

For a continuous mass distribution on a set S

$$\bar{x} = \int_S x \ dm \Big/ \int_S dm \ , \quad \bar{y} = \int_S y \ dm \Big/ \int_S dm \ , \quad \bar{z} = \int_S z \ dm \Big/ \int_S dm$$

$$I = \int_S r^2 \ dm \ , \quad r = r(x, y, z) \ .$$

$$\sigma = \sqrt{I/m} \qquad m = \int_S dm$$

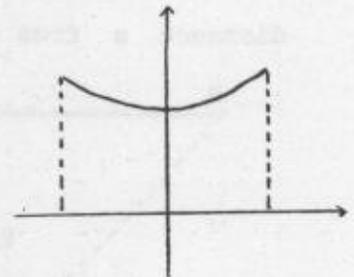where $dm = \rho d\ell$ for curves,

$dm = \rho da$ for surfaces and

$dm = \rho d\mu_3$ for solids and

$\rho = \rho(x, y, z)$ is the density of the mass distribution.

Example 2: Find the centroid of a thin uniform wire bent in the shape of the curve $y = \frac{1}{2}(e^x + e^{-x})$, $-1 \le x \le 1$

$$\gamma : \begin{cases} x = t \\ \\ y = \cosh t \end{cases} \qquad t \in [-1, 1]$$

$$\bar{x} = \int_\gamma x\,d\ell \Big/ \int_\gamma d\ell \qquad \qquad \underset{\sim}{v} = (x'(t),\ y'(t))$$

$$= (1,\ \sinh t)$$

$$= \int_{-1}^{1} t \cosh t\,dt \Big/ \int_{-1}^{1} \cosh t\,dt \qquad \therefore\ |\underset{\sim}{v}| = \sqrt{1 + \sinh^2 t} = \cosh t$$

$$= 0 \qquad \qquad d\ell = \cosh t\,dt$$

$$\bar{y} = \int_\gamma y\,d\ell \Big/ \int_\gamma d\ell = \int_{-1}^{1} \cosh^2 t\,dt \Big/ \int_{-1}^{1} \cosh t\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} (1 + \cosh 2t)\,dt \Big/ \int_{-1}^{1} \cosh t\,dt$$

$$= \frac{\frac{1}{2} + \frac{1}{4}\sinh 2t \Big|_{-1}^{1}}{\sinh t \Big|_{-1}^{1}} = \frac{1 + \frac{1}{2}\sinh 2}{2\sinh 1} = (e - e^{-1})^{-1} + \frac{1}{4}(e + e^{-1})$$

A point mass M at $(0, 0, 0)$ attracts a point mass m at (x, y, z) the force being given by $\underset{\sim}{F} = -\dfrac{k\,m\,M}{|\underset{\sim}{r}|^3}\underset{\sim}{r}$ where $\underset{\sim}{r} = (x, y, z)$; k is a universal constant. The <u>field</u> at (x, y, z) due to a body of gravitational material is defined to be the <u>force</u> exerted by the body on a unit point mass placed at (x, y, z) .

<u>Example 3</u>: Find the force exerted on a unit mass placed symmetrically at a distance a from a straight wire of uniform density and length $2R$.

$$\underset{\sim}{F} = (F_1,\ F_2)$$

$$r = |\underset{\sim}{r}|$$

$$F_1 = 0$$
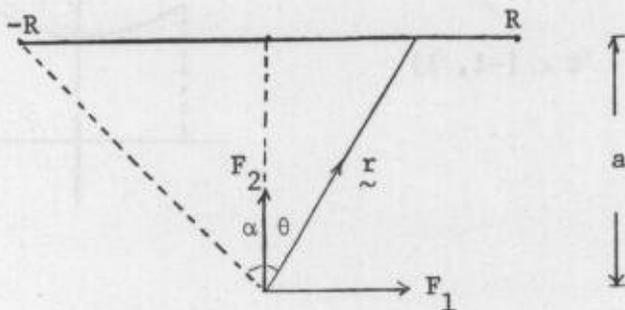
$$F_2 = \int_\gamma \frac{k \cos \theta}{r^2} \rho \, d\ell = ak\rho \int_\gamma \frac{1}{r^3} \, d\ell = ak\rho \int_{-R}^{R} \frac{1}{(a^2 + t^2)^{3/2}} \, dt$$

$$= \frac{k\rho}{a} \int_{-\tan^{-1}(\frac{R}{a})}^{\tan^{-1}(\frac{R}{a})} \cos u \, du \qquad\qquad t = a \tan u$$

$$= \frac{2k\rho}{a} \sin \left(\tan^{-1} \frac{R}{a}\right) = \frac{2k\rho}{a} \sin \alpha$$

The force due to an infinite wire (whatever that is) is $\dfrac{2k\rho}{a}$ along the axis of symmetry.

Example 4: Find the gravitational field due to a thin uniform spherical shell σ of radius 1 at a point P at distance a from the centre

$$\sigma : \begin{cases} x = \sin \phi \cos \theta, & 0 \le \theta \le 2\pi \quad d\sigma = |\underline{n}(\theta, \phi)| \, d\theta \, d\phi \\ y = \sin \phi \sin \theta, & 0 \le \phi \le \pi \qquad = \sin \phi \, d\theta \, d\phi \\ z = \cos \phi \end{cases}$$
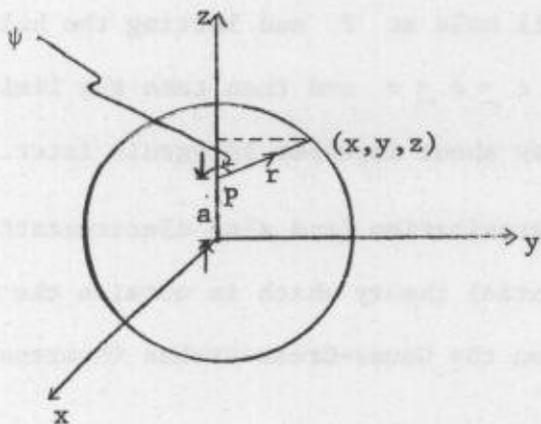
$$F = (F_1, F_2, F_3)$$

$$F_1 = F_2 = 0$$

$$F_3 = \int_\sigma \frac{k \cos \psi}{r^2} \rho \, da$$

$$= k\rho \int_\sigma \frac{z-a}{r^3} \, da$$

$$= k\rho \int_0^\pi \int_0^{2\pi} \frac{(\cos \phi - a)\sin \phi \, d\theta \, d\phi}{[\sin^2\phi + (\cos \phi - a)^2]^{3/2}}$$

$$= 2\pi k\rho \int_0^\pi \frac{(\cos \phi - a)\sin \phi \, d\phi}{[1 + a^2 - 2a \cos \phi]^{3/2}}$$

$$= \frac{\pi k\rho}{2a^2} \int_{(1-a)^2}^{(1+a)^2} \frac{1}{u^{3/2}}(1 - a^2 - u)du \ , \quad u = 1 + a^2 - 2a \cos \phi$$

$$= \frac{\pi k\rho}{a^2} \left[(1-a^2)\left(\frac{1}{|1-a|} - \frac{1}{|1+a|}\right) - (|1+a| - |1-a|) \right] \ , \quad a \neq 1 \ .$$

$$\therefore \quad F_3 = \begin{cases} 0 & , \quad 0 \leq a < 1 \ , \\ -\dfrac{2\pi k\rho}{a^2} & , \quad a = 1 \ , \\ -\dfrac{4\pi k\rho}{a^2} & , \quad a > 1 \ . \end{cases}$$

Strictly speaking we are dealing with an improper integral when $a = 1$ in that the integrand is unbounded in that case (P is on the surface). However we get the result we have obtained formally when $a = 1$ by computing the field for the surface with a small hole at P and letting the hole shrink (i.e. replace $0 \leq \phi \leq \pi$ by $\epsilon \leq \phi \leq \pi$ and then take the limit as $\epsilon \to 0 +$). We shall have more to say about improper integrals later.

A systematic treatment of gravitation (and also electrostatics and hydrodynamics) is best based on potential theory which is outside the scope of this course. It depends heavily on the Gauss-Green-Stokes theorems discussed in the next section.

Example 5: A line segment is divided at random into 3 parts. What is the probability they form a triangle?

$$x \geq 0$$
$$y \geq 0$$
$$0 \leq x + y \leq a$$
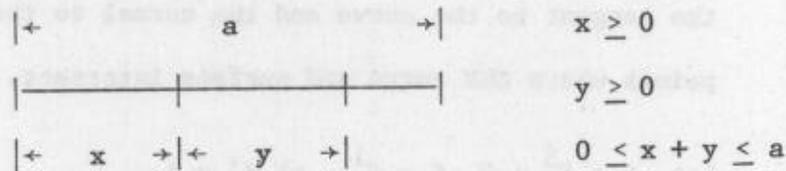
Let Δ be the set of all such points $(x,y) \in R^2$
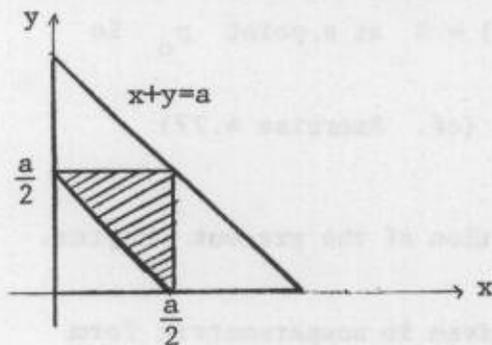
Favourable cases: Δ_1

$$x + y \geq a - x - y$$
$$a - x \geq x$$
$$a - y \geq y$$

i.e. Δ_1 is the triangle bounded by

$$x + y = \frac{a}{2}, \quad x = \frac{a}{2}, \quad y = \frac{a}{2}$$

$$\mu_2(\Delta) = \frac{1}{2}a^2 \quad \mu_2(\Delta_1) = \frac{1}{8}a^2$$

Therefore if (x,y) is chosen at random from Δ the probability that it comes from Δ_1 is

$$\mu_2(\Delta_1)/\mu_2(\Delta) = \frac{1}{4}.$$

Exercises:

5.21: The angle θ between two vectors p, q in R^n is given by $\cos \theta = p \cdot q/|p||q|$. Find the angle between the curves

$$(x, y, z) = (t, 2t, t^2) \qquad t \in R$$

$$(x, y, z) = (s^2, 1-s, 2-s^2) \qquad s \in R$$

where they intersect.

5.22: Sketch the curve $(x, y, z) = (\cos \pi t, \sin \pi t, t)$ and the surface $(x, y, z) = (v \cos \pi u, v \sin \pi u, v)$ and find the angle between the tangent to the curve and the normal to the surface at the points where the curve and surface intersect.

5.23: Let $f : R^3 \to R$, $f \in C^1$, rk $f' = 1$.

One can give a fairly convincing geometric argument that the normal to the surface $f(p) = 0$ at a point p_0 is

$$\nabla f(p_0) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)_{p_0} \quad (\text{cf. Exercise 4.27}).$$

Prove this from the definition of the present chapter.

5.24: Consider a curve in R^3 given in nonparametric form $f_1(x, y, z) = 0$, $f_2(x, y, z) = 0$. Prove that the tangent to the curve at a point p_0 has direction $\left(\dfrac{\partial (f_1, f_2)}{\partial (y,z)}, \dfrac{\partial (f_1, f_2)}{\partial (z,x)}, \dfrac{\partial (f_1, f_2)}{\partial (x,y)} \right)_{p_0}$.

A good geometric argument may be based on Exercise 5.20 or use the implicit function theorem. You may assume

$$\frac{\partial (f_1, f_2)}{\partial (x,y)}(p_0) \neq 0.$$

5.25: Find the surface area of the section of the paraboloid $z = x^2 + y^2$ for which $2 \leq x^2 + y^2 \leq 6$.

5.26: Show that a hemispherical surface of radius a has its centroid on its axis of symmetry at a distance $\frac{a}{2}$ from its base, and a solid hemisphere has its centroid at a distance $\frac{3a}{8}$ from its base.

5.27:　　Show that the radius of gyration about the z axis of the uniform solid bounded by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ is $\sqrt{\dfrac{a^2 + b^2}{5}}$.

5.28:　　Show that the centroid of an arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is $(a\pi , \dfrac{4}{3} a)$, $(0 \le \theta \le 2\pi)$.

5.29: (a) (Pappus' Theorems) If a plane area (curve) of area a (length ℓ) is rotated through an angle θ about a line in its plane not crossing the area (curve) then the volume (area) of the solid (surface) generated is the product of $a(\ell)$ and the length of the path of the centroid of the area (curve) .

(b) Derive the formula obtained for the area of the surface of a torus by using Pappus' Theorem.

(c) What is the formula for the volume of a torus?

[Pappus' Theorems are related to Guldin's Formula for the area (volume) of a surface (solid) swept out by a moving line segment (plane region) of variable length (area). This formula furnishes the theoretical basis for Amsler's planimeter, an instrument for measuring plane areas. Guldin's Formula is discussed in the book of Courant (Vol II, p. 294 et seq.)] .

5.30: (a) Find the centroid of the triangular region bounded by the lines

$y = \dfrac{a}{b} x$, $y = 0$, $x = h$.

(b) Find volume and lateral surface area of a right circular cone of base radius a and height h by direct integration and by using Pappus' Theorems.

5.31: A sphere of radius a has volume $\frac{4}{3}\pi a^3$ and surface area $4\pi a^2$. Use this and Pappus' Theorems to locate the centroids of a semi-circular plate, a quadrant, a semicircular wire, a quarter circle, all of radius a.

5.32: The density of a semicircular lamina at any point P is proportional to the square of the distance of P from the centre of the circle. Find the centre of mass of the lamina. [On the axis of symmetry at a distance $\frac{8a}{5\pi}$ from the straight edge.]

5.33: Given a hemispherical shell of density 1 having inner radius a and outer radius b find (a) its centroid, (b) its moment of inertia about the axis of symmetry, (c) its moment of inertia about a diameter of the base.

[(a) $(3(b^4 - a^4)/8(b^3 - a^3), 0, 0)$, (b) $4\pi(b^5 - a^5)/15$

(c) $4\pi(b^5 - a^5)/15$] .

5.34: Show that $M\frac{3a^2}{10}$ is the moment of inertia about the axis of symmetry of a uniform right circular cone of base radius a and mass M.

5.35: (Parallel Axes Theorem): Show that $I = I_G + M h^2$ where I is the moment of inertia of a body about a given axis, I_G is the moment of inertia about a parallel axis through the centroid of the body, M is its mass and h is the distance between the two axes.

5.36: If $a > b$ then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1 \quad , \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$$

are the equations of a <u>prolate</u> <u>spheroid</u> and an <u>oblate</u> <u>spheroid</u> respectively. They are generated by rotating the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{about its major and minor axes.} \quad \text{Show that their}$$

surface area is respectively

$$2\pi b^2 + (2\pi ab \; \sin^{-1}\epsilon)/\epsilon \quad , \quad 2\pi a^2 + (\pi b^2 \; \log\{\tfrac{1+\epsilon}{1-\epsilon}\})/\epsilon$$

where the eccentricity ϵ of the ellipse is defined by $b^2 = a^2(1-\epsilon^2)$.

5.37: Show that the gravitational attraction due to a uniform thin rod AB at an external point P is $\frac{2k\rho}{p} \sin \frac{1}{2} \alpha$ where ρ is the density, k is the gravitational constant, p is the perpendicular distance of P from the rod and α is the angle subtended by the rod at P.

5.38: Consider the right semicircular cylinder

$$D = \{(x, y, z) : x^2 + y^2 \le a^2 , 0 \le y , 0 \le z \le h\}$$

Show that the y-component of the gravitational attraction at $(0, 0, 0)$ due to a uniform solid mass distribution in the form D (density 1) is

$$2 \, k \, h \, \log[\frac{a + \sqrt{a^2 + h^2}}{h}]$$

[Express the force as an integral over D and evaluate the integral using cylindrical coordinates

$x = u \cos v, \; y = u \sin v, \; z = w, \; 0 \le u \le a, \; 0 \le v \le \pi, \; 0 \le w \le h$] .

5.39:   Show that a uniform spherical body of radius R and density ρ attracts a unit point mass placed at a distance a from its centre with a force of magnitude $\frac{4}{3} \pi R^3 k \rho / a^2$, $a > R$ and $\frac{4}{3} \pi a k \rho$, $a \leq R$.

5.40:   A solid uniform cylinder has a given volume. Show that the attraction at the centre of one of the circular ends is a maximum when the ratio of the height of the cylinder to the radius of the end is $(9 - \sqrt{17})/8$.

5.41:   The vertical angle of a solid uniform cone is 90°. Prove that the ratio of the attraction at the centre of the base to that at the vertex is approximately 1.29.

5.42:   Two numbers x and y are chosen at random between 0 and 4. Show that the probability that their product xy is less than 4 is $\frac{1}{4} + \log \sqrt{2}$.

5.43:   Show that the mean distance of the points of a circular area (radius a) from the end of a diameter is $32a/9\pi$.

5.44:   An interval a is divided into three subintervals. What is the mean value of the length of the shortest interval? $[\frac{a}{9}]$.

5.45:   The mean distance of the points on the surface of a sphere of radius a from a point on the surface is $\frac{4a}{3}$.

5.46:   Show that the mean value of one of n positive numbers whose sum does not exceed 1 is $\frac{1}{n+1}$.

5.47:   Prove that the normals to a smooth surface in R^3 intersect the z-axis if and only if it is a surface of revolution.

5.48: A (1-1) map is said to be 'conformal' if the angle between any two intersecting curves is the same as the angle between their images.

(a) Inversion about the unit circle in R^2 is defined by

$$u = \frac{x}{x^2 + y^2} \; , \quad v = \frac{y}{x^2 + y^2}$$

Prove that this map is conformal.

(b) Prove that the image of any circle or straight line under inversion is either a circle or a straight line.

(c) Find the Jacobian of the inversion.

5.49: Prove that in a curvilinear triangle formed by three circles which pass through a common point O the sum of the interior angles at the vertices is π (O is not a vertex of the triangle).

5.50: (a) A map of the plane given by

$$u = \phi(x,y) \quad , \quad v = \psi(x,y)$$

is locally (1-1) and conformal if

$$\phi_x = \psi_y \, , \, -\phi_y = \psi_x \quad \text{(Cauchy-Riemann equations)}$$

and ϕ_x , ϕ_y are not both zero simultaneously.

(b) $u = e^x \cos y \quad v = e^x \sin y$ is conformal. Draw the images of lines $x = $ const. and $y = $ const.

(c) $u = \frac{1}{2}(x + \frac{x}{x^2 + y^2}) \; , \quad v = \frac{1}{2}(y - \frac{y}{x^2 + y^2})$ is conformal and maps straight lines through $(0,0)$ and circles $x^2 + y^2 = k$ into confocal conics $\dfrac{u^2}{c + \frac{1}{2}} + \dfrac{v^2}{c - \frac{1}{2}} = 1$.

5.51:   Inversion in three dimensions is defined by

$$u = \frac{x}{x^2 + y^2 + z^2}, \quad v = \frac{y}{x^2 + y^2 + z^2}, \quad w = \frac{z}{x^2 + y^2 + z^2}.$$

(a) The angle between normals to surfaces is preserved.

(b) Spheres are transformed into either spheres or planes.

5.52:   Prove that the tangent plane to the surface $xyz = a^3$ at any point of the surface forms with the three coordinate planes a tetrahedron of constant volume.

THE TRUTH ABOUT GAUSS, GREEN AND STOKES

There are integrals associated with curves and surfaces other than those discussed hitherto. For example if the position of a particle in space at any time $t \in R$ is given by $\gamma(t) = (x(t), y(t), z(t))$ and is subject to a force $\underset{\sim}{F} = \underset{\sim}{F}(x, y, z)$ at each point (x, y, z) it is reasonable to define the work done between time $t=a$ and $t=b$ as

$$W = \int_a^b \underset{\sim}{F} \cdot \underset{\sim}{v} \qquad\qquad \underset{\sim}{F} = (F_1, F_2, F_3)$$

$$\underset{\sim}{v} = (x_1', x_2', x_3')$$

$$= \int_a^b \{F_1(\gamma(t))x'(t) + F_2(\gamma(t))y'(t) + F_3(\gamma(t))z'(t)\}\, dt$$

$$\overset{def}{=} \int_\gamma F_1 dx + F_2 dy + F_3 dz$$

As a further example consider a fluid flow in space the velocity at any point (x, y, z) being $\underset{\sim}{V}(x, y, z)$. Let $\sigma : \sigma(u,v) = (x(u,v), y(u,v), z(u,v))$, $(u,v) \in D \subset R^2$ be a surface in the fluid. The volume of fluid crossing the surface per unit time is

$$\int_D \underset{\sim}{V} \cdot \underset{\sim}{n}$$

$$= \int_D \{V_1(\sigma(u,v)) \frac{\partial (y,z)}{\partial (u,v)} + V_2(\sigma(u,v)) \frac{\partial (z,x)}{\partial (u,v)} + V_3(\sigma(u,v)) \frac{\partial (x,y)}{\partial (u,v)}\}\, du\, dv$$

$$\overset{def}{=} \int_\sigma V_1 dy\, dz + V_2 dz\, dx + V_3 dx\, dy .$$

Notice however that these integrals, while they are invariant with respect to equivalent parametrizations of the curve or surface, they change

sign when evaluated with respect to parametrizations which reverse the direction of the tangent to the curve or the normal to the surface. Thus we must consider the concept of orientation. Our discussion of orientation involves a certain amount of handwaving; in fact an adequate understanding for this course can be obtained from a classroom discussion with pictures and examples. The subject is usually covered in detail in Mathematics 422.

Orientation: We endow lines, planes and three dimensional Euclidean space with an orientation when we adopt a coordinate system in these objects. We have two essential choices in each case:

(a)
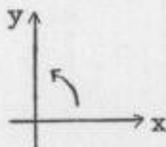
$R: \longrightarrow$ $R^2:$ $R^3:$
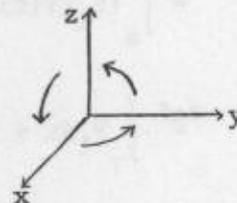
or (b)
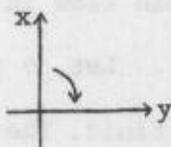
$R: \longleftarrow$ $R^2:$ $R^3:$
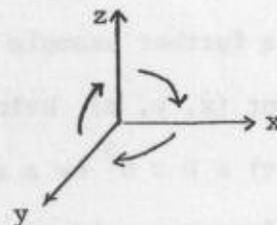
We shall adopt the orientations of R, R^2, R^3 sketched in (a) as standard. All cartesian coordinate systems which can be obtained one from the other by a translation and rotation (affine transformation with Jacobian +1) are considered to endow the space with the same orientation. To obtain a system of type (b) from one of type (a) requires an affine transformation with Jacobian −1; these are considered opposite orientations. In general an orientation is achieved in R^n by choosing a set of n independent axes, specifying a positive direction on each and then listing them in some order.

Similarly given a k-dimensional segment S in R^n, $0 < k \leq n$ (cf. page 220), a specific parametrization ϕ of S, which is a coordinate system for S (cf. page 223), is also an orientation for S.

$$D \subset R^k \xrightarrow[(1-1)]{\phi} S \subset R^n$$

k = 1 :

k = 2 :

k = 3 :
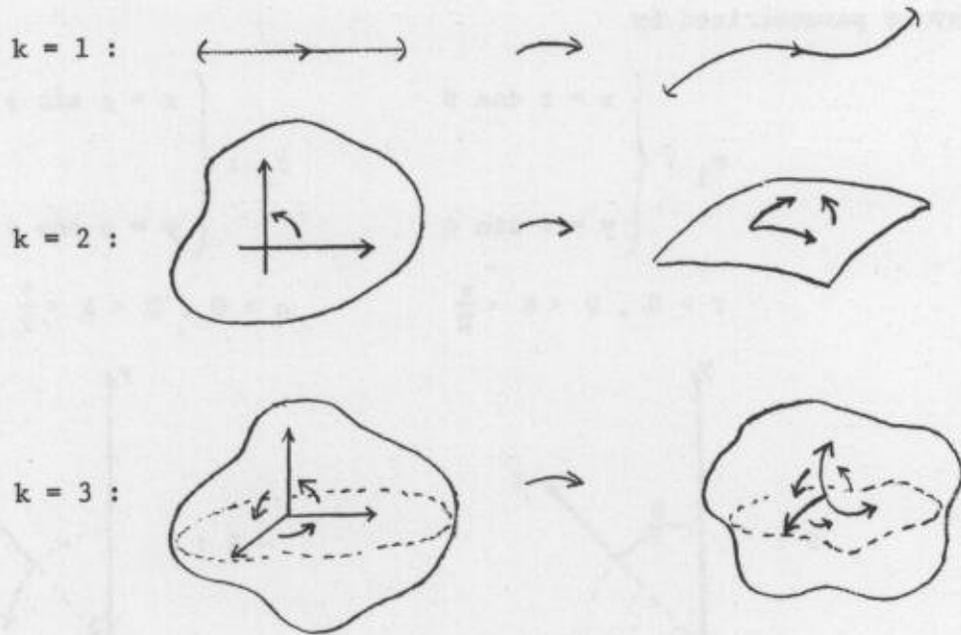
So when we speak of an oriented segment S we are really considering a pair $\{S, \phi\}$ i.e. the segment S and a parametrization ϕ or, to be even more boring about it, the segment S and an equivalence class of parametrically equivalent coordinate systems. For example $\gamma_1(t) = (\cos t , \sin t)$, $t \in (0, 2\pi)$, $\gamma_2(t) = (\cos t , -\sin t)$, $t \in (0, 2\pi)$ are parametrizations of the same 1-dimensional segment in R^2 :

$\gamma_1 :$

$\gamma_2 :$
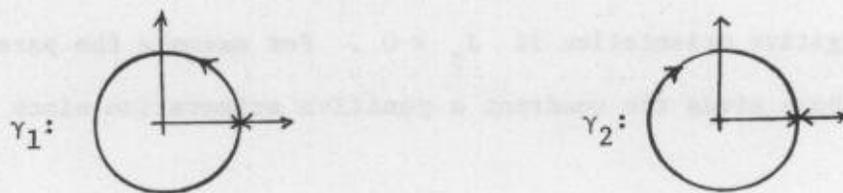
but give it opposite orientations. All curves parametrically equivalent to γ_1 have the same orientation (direction cf. page 266) but $\gamma_2(t) = \gamma_1(f(t))$ where $f(t) = 2\pi - t$, $f'(t) = -1$ (parametric equivalence would require $f'(t) > 0$).

As a further example the segment $S = \{(x,y) \in R^2 : x > 0, y > 0\}$ may be parametrized by

$$\sigma_1 : \begin{cases} x = r \cos \theta \\ \\ y = r \sin \theta \end{cases} \qquad \sigma_2 : \begin{cases} x = \rho \sin \phi \\ \\ y = \rho \cos \phi \end{cases}$$

$$r > 0 , \; 0 < \theta < \frac{\pi}{2} \qquad\qquad \rho > 0 , \; 0 < \phi < \frac{\pi}{2}$$
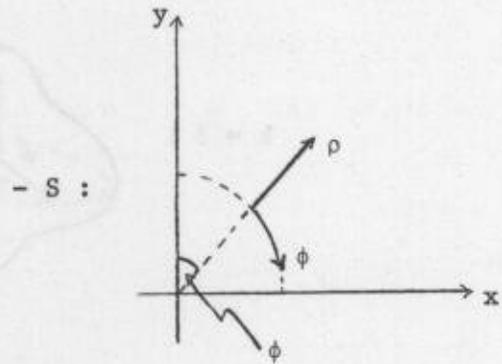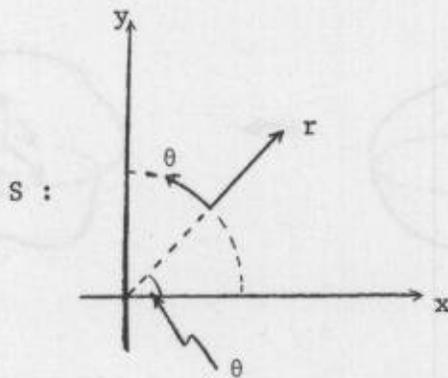
However they give it opposite orientations. So if we choose to consider the segment with the orientation (r, θ) and to denote it by S then we denote the segment with the orientation (ρ, ϕ) by $- S$.

In general having adopted a standard orientation for R^n we can define positive and negative orientation for n-dimensional segments in R^n. A parametrization ϕ gives such a segment a positive orientation if $J_\phi > 0$ and a negative orientation if $J_\phi < 0$. For example the parametrization (r, θ) above gives the quadrant a positive orientation since $\frac{\partial (x,y)}{\partial (r,\theta)} = r > 0$

and (ρ, ϕ) gives it a negative orientation since $\frac{\partial (x,y)}{\partial (\rho,\phi)} = -\rho < 0$. The parametrizations (θ, r), (ϕ, ρ) give negative and positive orientations respectively. In R a parametrized interval is positively or negatively oriented if its parametrization is consistent with or opposite to that of the line i.e. if $a < b$ then $\gamma_1(t) = a + t(b - a)$ gives the interval (a, b) a positive orientation while $\gamma_2(t) = b + t(a - b)$ gives it a negative orientation.

Geometrically, for curves in R^3, it is convenient to think of orientation as specifying a direction on the curve. For a surface it amounts to specifying a normal consistently drawn on one side of the surface

$$\underset{\sim}{n} = \underset{\sim}{u} \times \underset{\sim}{v}$$
$$-\underset{\sim}{n} = \underset{\sim}{v} \times \underset{\sim}{u}$$
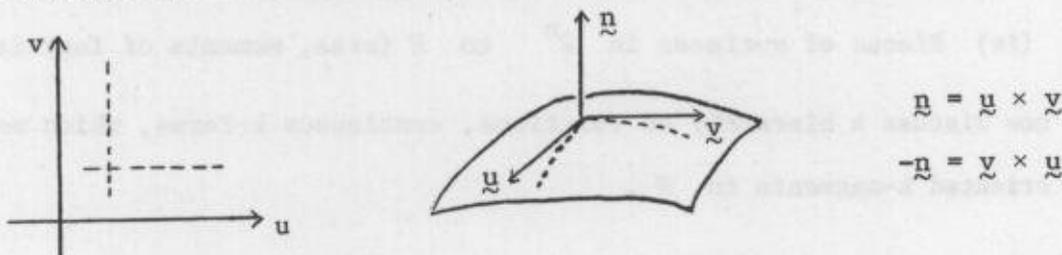
Opposite orientations give normals on opposite sides of the surface.

Finally 0-dimensional segments (points) are oriented by labelling them " + " or " - ".

To summarize, a k-dimensional segment inherits, through its parametrization, an orientation from the standard orientation of the underlying parameter space R^k.

General manifolds (unlike segments) cannot always be oriented: for example we can distinguish between two sides of a cylinder or a sphere so they can be oriented but this is not the case for the Möbius band (an

obscure rock group).
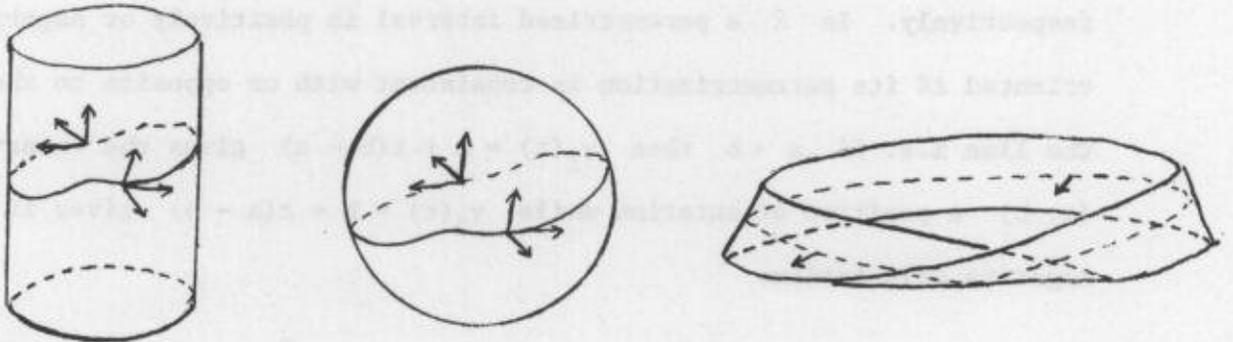
Differential Forms:

We have already encountered functions which map:

(i) points in R^n to points in R^m (f : $R^n \to R^m$)

(ii) Jordan measurable subsets of R^n to R (Jordan content, integrals)

(iii) Pieces of curves in R^n to R (length, moments of inertia, etc.)

(iv) Pieces of surfaces in R^n to R (area, moments of inertia, etc.)

We now discuss a hierarchy of functions, continuous k-forms, which map pieces of oriented k-segments to R .

Forms in R :

0-forms: $\omega = A(x)$

A is a continuous real valued function on R. If τ is a positively oriented 0-segment i.e. a point x labeled " + " then

$$\omega(\tau) \overset{\text{def}}{=} A(x)$$

$$\omega(-\tau) \overset{\text{def}}{=} -A(x) = -\omega(\tau)$$

where $-\tau$ denotes the point x labeled " - " .

1-forms: $\omega = A(x)dx$ $\qquad\qquad A \in C$

Let $\gamma[a, b]$ be an interval in R oriented by the parametrization
$\gamma : R \to R$ where $\gamma \in C^1$ and $rk \ \gamma' = 1$ (i.e. $\gamma' > 0$ or $\gamma' < 0$) then

$$\omega(\gamma[a, b]) = \int_{\gamma(a)}^{\gamma(b)} A(x)dx$$

$$\stackrel{def}{=} \int_a^b A(\gamma(t)) \gamma'(t) \ dt$$

thus

$$\omega(\gamma[a,b]) = \begin{cases} \displaystyle\int_{\gamma[a,b]} A & \text{if } \gamma' > 0 \\[2em] -\displaystyle\int_{\gamma[a,b]} A & \text{if } \gamma' < 0 \end{cases}$$

$$\begin{array}{c} \gamma' > 0 \\ \bullet\!\!\longrightarrow\!\!\bullet \\ \gamma(a) \qquad \gamma(b) \end{array}$$

$$\begin{array}{c} \gamma' < 0 \\ \bullet\!\!\longleftarrow\!\!\bullet \\ \gamma(b) \qquad \gamma(a) \end{array}$$

i.e. $\omega(I) = \displaystyle\int_I A$ if I is a positively oriented interval

and $\omega(I) = -\displaystyle\int_I A$ if I is negatively oriented

Note that when we adopted the convention $\displaystyle\int_a^b = -\int_b^a$ previously we were

really not just considering $\displaystyle\int_{[a,b]}$ but were taking orientation into account
also.

Forms in R^2 :

0-forms: $\omega = A(x,y)$ $\qquad\qquad A \in C$

$$\omega(\tau) \stackrel{def}{=} A(x,y) \qquad\qquad \tau : \stackrel{+}{\bullet} (x,y)$$

$$\omega(-\tau) \stackrel{def}{=} -A(x,y) = -\omega(\tau) \qquad -\tau : \stackrel{-}{\bullet} (x,y)$$

<u>1-forms:</u> $\omega = A(x,y)dx + B(x,y)dy$, $A, B \in C$

Let γ be a smooth curve in R^2 (i.e. $\gamma : R \to R^2$ $\gamma \in C^1$, rk $\gamma' = 1$) and γ is (1-1). Then if $\gamma :$ $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

$$\omega(\gamma[a,b]) = \int_{\gamma(a)}^{\gamma(b)} A(x,y)dx + B(x,y)dy$$
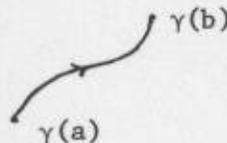
$$\overset{def}{=} \int_a^b \{A(x(t), y(t))x'(t) + B(x(t), y(t))y'(t)\} \, dt$$

<u>2-forms:</u> $\omega = A(x,y)dx \, dy$

Let D be a reasonable set in R^2 (D Jordan measurable, $D = \overline{D^\circ}$) and $\sigma[D]$ a set in R^2 oriented by the parametrization $\sigma : R^2 \to R^2$ where $\sigma \in C^1$, rk $\sigma' = 2$, σ(1-1) , then if

$$\sigma : \begin{cases} x = x(u,v) \\ \\ y = y(u,v) \end{cases} , \quad (u,v) \in D , \frac{\partial(x,y)}{\partial(u,v)} > 0 \text{ or } \frac{\partial(x,y)}{\partial(u,v)} < 0$$

$$\omega(\sigma[D]) = \int_{\sigma[D]} A(x,y)dx \, dy$$

$$\overset{def}{=} \int_D A(x(u,v),y(u,v)) \frac{\partial(x,y)}{\partial(u,v)} \, du \, dv \quad (*)$$
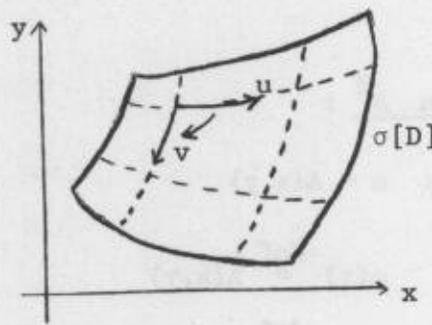
$$J_\sigma = \frac{\partial(x,y)}{\partial(u,v)} > 0 \qquad\qquad J_\sigma = \frac{\partial(x,y)}{\partial(u,v)} < 0$$

In order that this definition be unambiguous we must adopt the convention that "du dv" means "$d\mu_2$" while "dv du" means "$-d\mu_2$" i.e. dv du = -du dv . With this convention $\frac{\partial(x,y)}{\partial(u,v)}$ du dv = $\frac{\partial(x,y)}{\partial(v,u)}$ dv du = J_σ $d\mu_2$. Thus if S is a reasonable set in R^2 then

$$\omega(S) = \int_S A = \int_S A \ dx \ dy \ , \ \text{if S is positively oriented,}$$

$$\omega(S) = -\int_S A = \int_S A \ dy \ dx, \ \text{if S is negatively oriented.}$$

N.B. This convention has nothing to do with Fubini's Theorem.

Forms in R^3 :

0-forms: $\omega = A(x, y, z)$ $\qquad\qquad A \in C$

1-forms: $\omega = A(x, y, z)dx + B(x, y, z)dy + C(x, y, z)dz$, A, B, C $\in$ C

$$\gamma : \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

$$\omega(\gamma[a,b]) = \int_{\gamma(a)}^{\gamma(b)} A \ dx + B \ dy + C \ dz$$

$$\overset{\text{def}}{=} \int_a^b \{A(\gamma(t))x'(t) + B(\gamma(t))y'(t) + C(\gamma(t))z'(t)\} \ dt$$

2-forms: $\omega = A(x,y,z)dy \ dz + B(x,y,z)dz \ dx + C(x,y,z)dx \ dy$, A, B, C $\in$ C

$$\sigma : \begin{cases} x = x(u,v) \\ y = y(u,v) \ , \ (u,v) \in D \\ z = z(u,v) \end{cases}$$

$$\omega(\sigma[D]) = \int_{\sigma[D]} A \ dy \ dz + B \ dz \ dx + C \ dx \ dy$$

$$\overset{\text{def}}{=} \int_{D} \{A(\sigma(u,v)) \frac{\partial (y,z)}{\partial (u,v)} + B(\sigma(u,v)) \frac{\partial (z,x)}{\partial (u,v)} + C(\sigma(u,v)) \frac{\partial (x,y)}{\partial (u,v)} \} \ du \ dv$$

We still have the

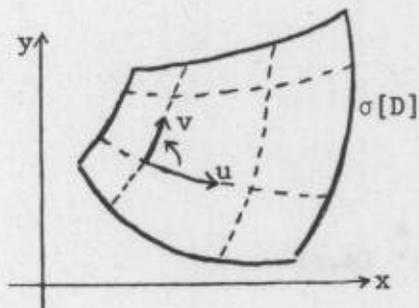convention that

$du \ dv = -dv \ du = d\mu_2$.

3-forms: $\omega = A(x, y, z)dx \ dy \ dz$
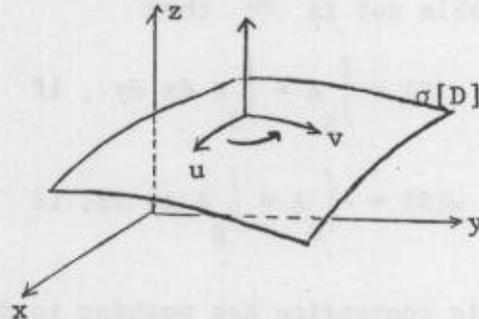
Let G be a reasonable set in R^3 (G Jordan measurable, $G = \overline{G^{\circ}}$)
and $\nu[G]$ a set in R^3 oriented by the parametrization $\nu : R^3 \to R^3$ where
$\nu \in C^1$, rk $\nu' = 3$, ν(1-1), then if

$$\nu : \begin{cases} x = x(u, v, w) \\ y = y(u, v, w) \ , \ (u, v, w) \in G \ , \ \frac{\partial (x,y,z)}{\partial (u,v,w)} > 0 \ \text{ or } \ \frac{\partial (x,y,z)}{\partial (u,v,w)} < 0 \\ z = z(u, v, w) \end{cases}$$

$$\omega(\nu[G]) = \int_{\nu(G)} A(x, y, z)dx \ dy \ dz$$

$$\overset{\text{def}}{=} \int_{G} A(x(u,v,w), \ y(u,v,w), \ z(u,v,w)) \frac{\partial (x,y,z)}{\partial (u,v,w)} \ du \ dv \ dw$$

$$\frac{\partial (x,y,z)}{\partial (u,v,w)} > 0 \qquad\qquad\qquad \frac{\partial (x,y,z)}{\partial (u,v,w)} < 0$$
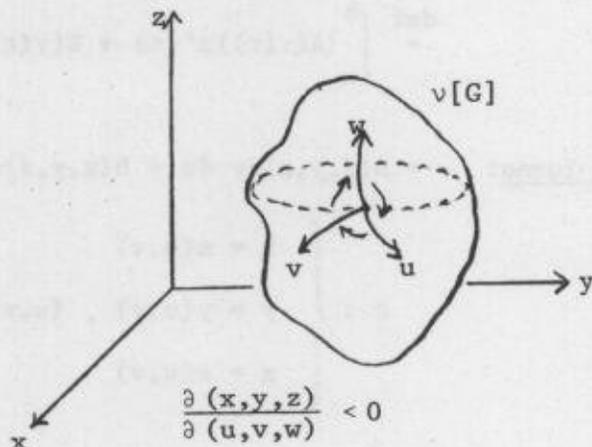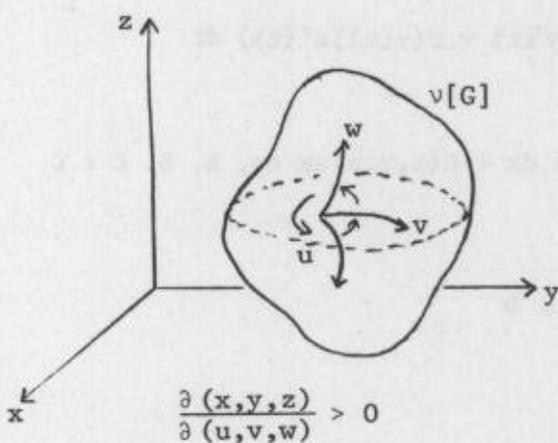
For consistency we must adopt the convention

$$du\ dv\ dw = dv\ dw\ du = dw\ du\ dv = -dv\ du\ dw = -dw\ dv\ du = -du\ dw\ dv = d\mu_3$$

Thus if V is a reasonable set in R^3 then

$$\omega(V) = \int_V A = \int_V A\ dx\ dy\ dz \text{ , if } V \text{ is positively oriented,}$$

$$\omega(V) = -\int_V A = \int_V A\ dy\ dx\ dz, \text{ if } V \text{ is negatively oriented.}$$

Remarks:

(a) Notice that, for all the forms introduced, changing the orientation of a set S simply changes the sign of $\omega(S)$, i.e. $\omega(-S) = -\omega(S)$.

(b) The particular parametrization used in evaluating $\omega(S)$ is not important. Two equivalent parametrizations give the same value for $\omega(S)$. We omit the details, which just involve the change of variable formula for integrals (Theorem 5.2).

Example 1: $\omega = x\ dx + xy\ dy$

(a)
$$\gamma_1 : \begin{cases} x = t \\ \\ y = t \end{cases} \quad 0 \le t \le 1$$

$\gamma_1[0,1]$
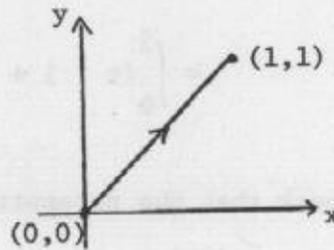
$$\omega(\gamma_1[0,1]) = \int_{\gamma_1(0)}^{\gamma_1(1)} x\ dx + xy\ dy$$

$$= \int_0^1 (t \cdot 1 + t^2 \cdot 1)dt = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

(b)
$$\gamma_2 : \begin{cases} x = 1 - t \\ \quad\quad\quad\quad 0 \le t \le 1 \\ y = 1 - t \end{cases}$$

$\gamma_2[0,1]$
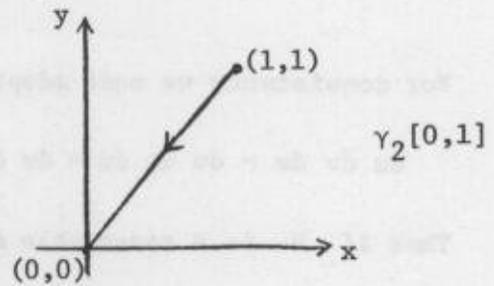
$$\omega(\gamma_2[0,1]) = \int_{\gamma_2(0)}^{\gamma_2(1)} x\, dx + xy\, dy$$

$$= \int_0^1 [(1-t)(-1) + (1-t)^2(-1)]\, dt$$

$$= -\frac{1}{2} - \frac{1}{3} = -\frac{5}{6}$$

Notice that $\gamma_2[0,1]$ is $\gamma_1[0,1]$ with the opposite orientation
i.e. $\gamma_2[0,1] = -\gamma_1[0,1]$.

(c)
$$\gamma : \begin{cases} x = t \\ \quad\quad\quad\quad 0 \le t \le 2 \\ y = t^2 \end{cases}$$

$\gamma[0,2]$
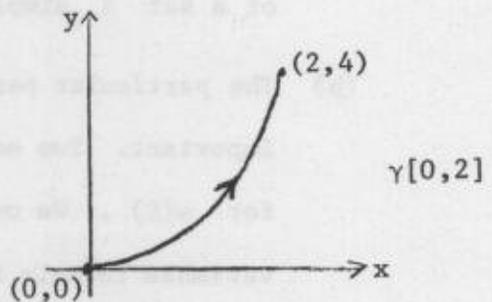
$$\omega(\gamma[0,2]) = \int_{\gamma(0)}^{\gamma(2)} x\, dx + xy\, dy$$

$$= \int_0^2 (t \cdot 1 + t^3 \cdot 2t)\, dt = 14\frac{4}{5}$$

Check that the parametrization $\gamma^*(t) = (t^2, t^4)\ 0 \le t \le \sqrt{2}$

gives the same result as example (c). Find a parametrization that reverses

the orientation and check that $\omega(-S) = -\omega(S)$.

Example 2: $\omega = dx\, dy$

(a)

$$\sigma_1 : \begin{cases} x = \sin \phi_1 \cos \theta_1 \\ y = \sin \phi_1 \sin \theta_1 \\ z = \cos \phi_1 \end{cases} \quad D_1 = \{(\phi_1, \theta_1) : \frac{\pi}{4} \le \phi_1 \le \frac{\pi}{2}, \ 0 \le \theta_1 \le \frac{\pi}{2}\}$$
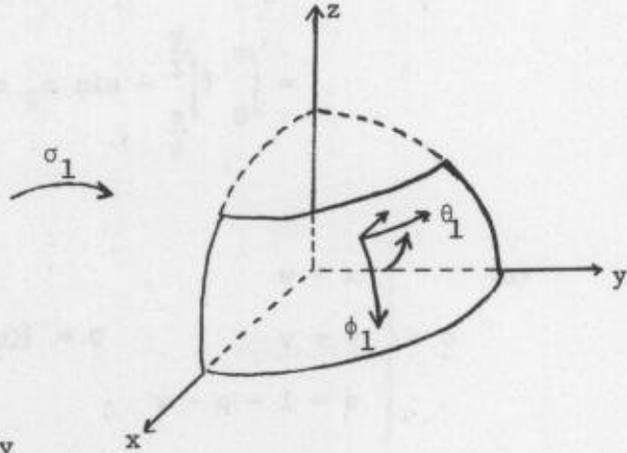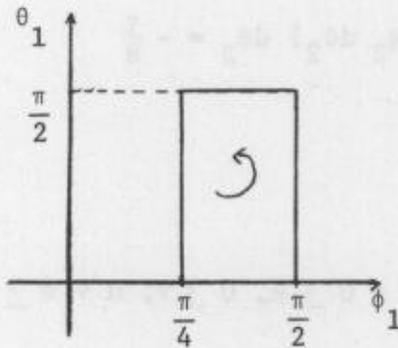
$$\omega(\sigma_1[D_1]) = \int_{\sigma_1[D_1]} dx\, dy$$

$$= \int_{D_1} \frac{\partial(x,y)}{\partial(\phi_1, \theta_1)}\, d\phi_1\, d\theta_1$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left\{ \int_0^{\frac{\pi}{2}} \sin \phi_1 \cos \phi_1\, d\phi_1 \right\} d\theta_1 = \frac{\pi}{8}$$

(b)

$$\sigma_2 : \begin{cases} x = \sin \phi_2 \cos \theta_2 \\ y = \sin \phi_2 \sin \theta_2 \\ z = \cos \phi_2 \end{cases} \quad D_2 = \{(\theta_2, \phi_2) : 0 \le \theta_2 \le \frac{\pi}{2}, \ \frac{\pi}{4} \le \phi_2 \le \frac{\pi}{2}\}$$
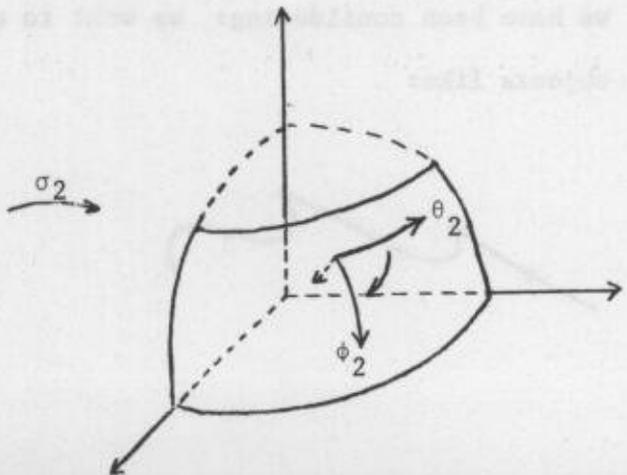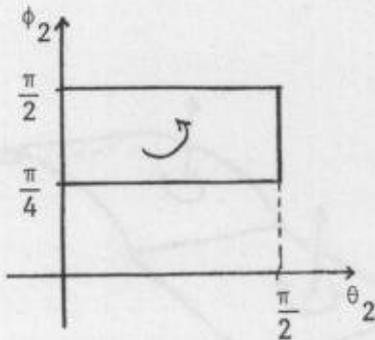
$$\omega(\sigma_2[D_2]) = \int_{\sigma_2[D_2]} dx \, dy$$

$$= \int_{D_2} \frac{\partial(x,y)}{\partial(\theta_2, \phi_2)} \, d\theta_2 \, d\phi_2$$

$$= \int_0^\pi \left\{ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \sin \phi_2 \cos \phi_2 \, d\theta_2 \right\} d\phi_2 = -\frac{\pi}{8}$$

(c) $\sigma : \begin{cases} x = u \\ y = v \\ z = 1 - u - v \end{cases}$ $D = \{(u,v) : 0 \leq u, \ 0 \leq v, \ u + v \leq 1\}$
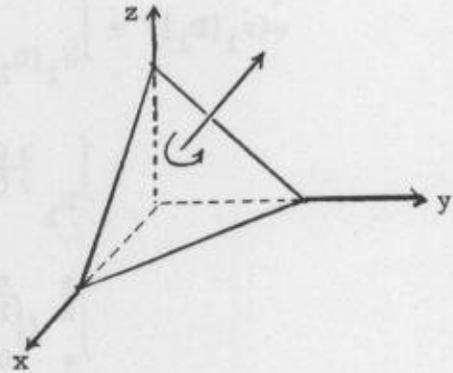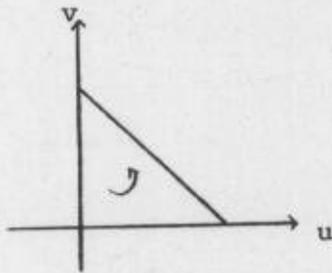
$$\omega(\sigma[D]) = \int_{\sigma[D]} dx \, dy$$

$$= \int_D \frac{\partial(x,y)}{\partial(u,v)} \, du \, dv = \int_D du \, dv = \frac{1}{2}$$

We now wish to relax the smoothness conditions on the parametrizations we have been considering: we want to extend the domains of forms to include objects like:
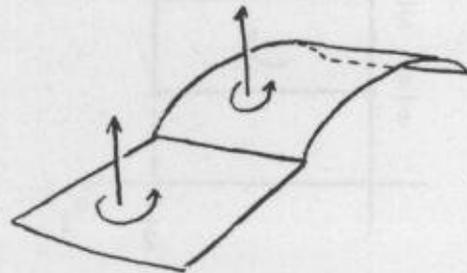
If $D \subset R^k$ and $S = \sigma[D]$ is a smooth k-surface in R^n oriented by $\sigma : R^k \to R^n$, we define $\partial S = \sigma[\partial D]$ to be the "boundary" or "edge" of S. It is not necessarily the actual boundary of S (at least in the topology of R^n); for example a smooth curve γ in R^2 is its own boundary but we have defined $\partial \gamma$ to mean the end points of γ. First observe that an orientation on a k-surface S can be thought to induce an orientation on ∂S in a natural way if ∂S is "nice" i.e. an orientable (k-1)-surface. For example:
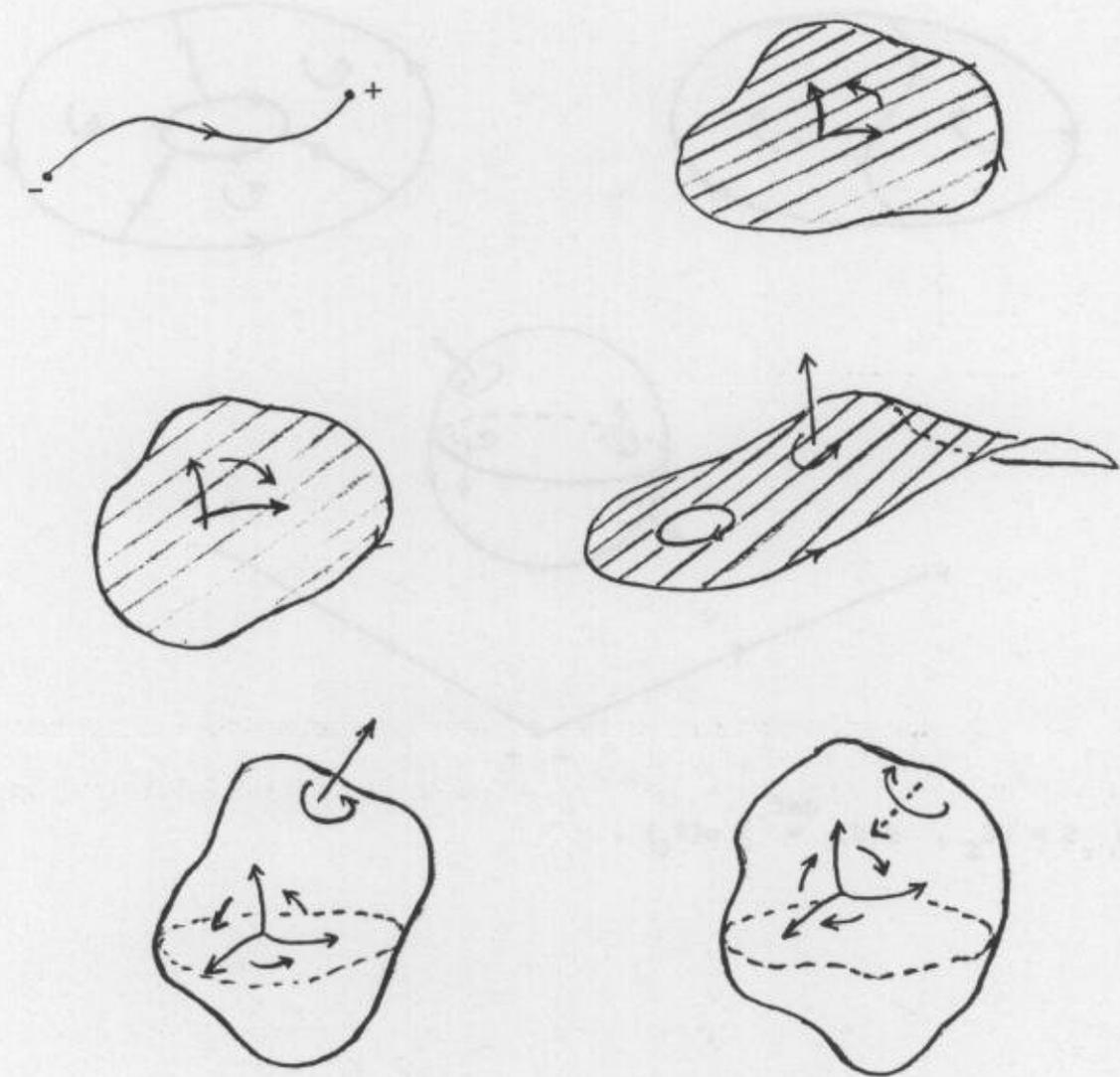
Then a piecewise smooth oriented k-surface is composed of finitely many oriented k-dimensional pieces S_i such that, if $i \neq j$, S_i and S_j have at most boundary segments in common and the orientation of $\partial S_i \cap \partial S_j$ as a subset of ∂S_i is opposite to its orientation as a subset of ∂S_j.
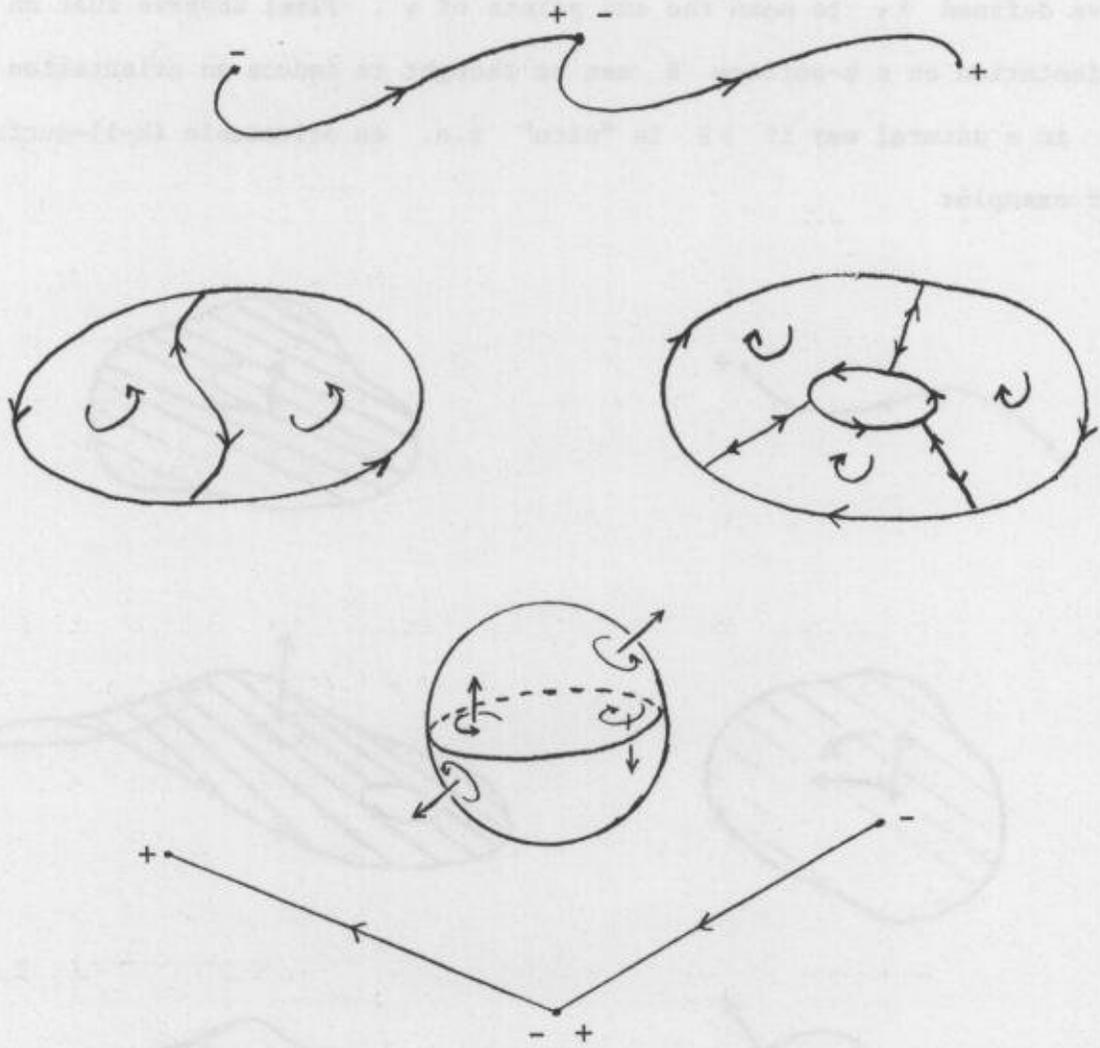
If $S = \bigcup_i S_i$, $\omega(S) \overset{\text{def}}{=} \sum_i \omega(S_i)$.

Example: $\omega = x\,dx + xy\,dy$

$\gamma = \gamma_1 + \gamma_2 + \gamma_3$

$\omega(\gamma) = \omega(\gamma_1) + \omega(\gamma_2) + \omega(\gamma_3)$

$= \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{2} = \dfrac{2}{3}$
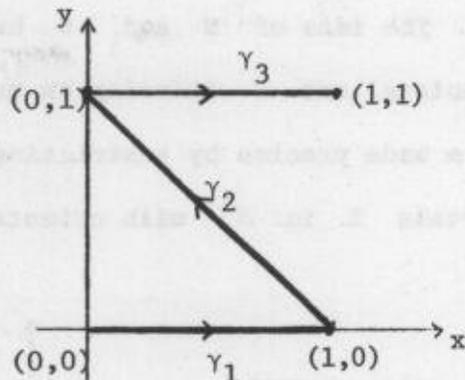
Exercises:

5.53: Make a Möbius band; check that if you try to divide it into two oriented surfaces with oriented boundaries then one of the common portions of the boundary has the same orientation as a subset of each surface. Colour your efforts with your crayons and give the result to the Chairman of the Mathematics Department. Don't say who suggested this.

5.54: You should now consider why Theorem 5.2 is more restrictive in R^n, $n \geq 2$ than in R^1; i.e. $\displaystyle\int_{\phi(a)}^{\phi(b)} f = \int_a^b (f \circ \phi)\phi'$ with very little restriction on ϕ and ϕ' while for $n \geq 2$ we require that ϕ be (1-1) and J_ϕ be of one sign

Note: The idea of S and ∂S being "nice" and an orientation in the k-dimensional set S inducing an orientation in the (k-1)-dimensional set ∂S may be made precise by restricting the domains of parameters to be oriented intervals I in R^k with oriented boundaries as follows:

k = 1   k = 2   k = 3
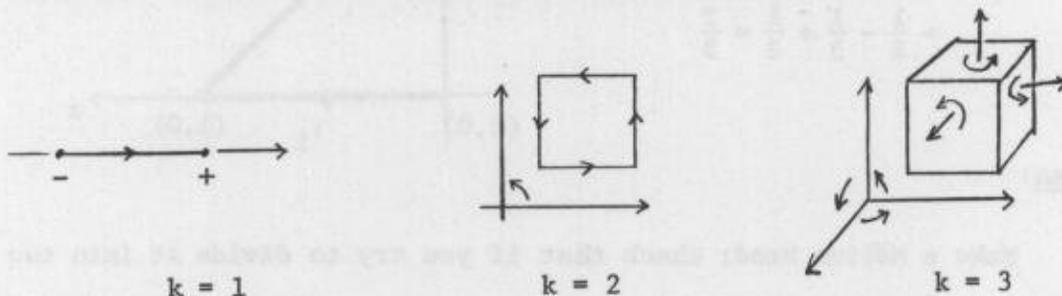
The parametrization of S then induces an orientation not only on S but also on ∂S (e.g. see Exercise 4.76, p. 237).

Algebra of forms:

Addition: We may define addition of k-forms e.g. in R^3

0-forms: ω = A(x, y, z)   α = B(x, y, z)

   ω + α = A(x, y, z) + B(x, y, z) = α + ω

1-forms: ω = A dx + B dy + C dz, α = L dx + M dy + N dz

   ω + α = (A + L)dx + (B + M)dy + (C + N)dz = α + ω

2-forms: ω = A dy dz + B dz dx + C dx dy, α = L dy dz + M dz dx + N dx dy

   ω + α = (A + L)dy dz + (B + M)dz dx + (C + N)dx dy = α + ω

3-forms: ω = A dx dy dz   α = L dx dy dz

   ω + α = (A + L)dx dy dz = α + ω

<u>Multiplication of two forms</u>: We may define multiplication of any two forms by the following rules:

    1. $(\omega_1 + \omega_2)\alpha = \omega_1\alpha + \omega_2\alpha$

    2. $dx\,dy = -dy\,dx$, $\quad dy\,dz = -dz\,dy$, $\quad dz\,dx = -dx\,dz$

    3. $dx\,dx = dy\,dy = dz\,dz = 0$

<u>Examples:</u>

    (a) $\omega = A$, $\alpha = L$

         $\omega\alpha = AL = LA = \alpha\omega$

    (b) $\omega = A$, $\alpha = L\,dx + M\,dy + N\,dz$

         $\omega\alpha = AL\,dx + AM\,dy + AN\,dz = LA\,dx + MA\,dy + NA\,dz = \alpha\omega$

    (c) $\omega = A$, $\alpha = L\,dy\,dz + M\,dz\,dx + N\,dx\,dy$

         $\omega\alpha = AL\,dy\,dz + AM\,dz\,dx + AN\,dx\,dy$

           $= LA\,dy\,dz + MA\,dz\,dx + NA\,dx\,dy = \alpha\omega$

    (d) $\omega = A$, $\alpha = L\,dx\,dy\,dz$

         $\omega\alpha = AL\,dx\,dy\,dz = LA\,dx\,dy\,dz = \alpha\omega$

    (e) $\omega = A\,dx + B\,dy + C\,dz$, $\alpha = L\,dx + M\,dy + N\,dz$

         $\omega\alpha = AL\,dx\,dx + AM\,dx\,dy + AN\,dx\,dz$

           $+ BL\,dy\,dx + BM\,dy\,dy + BN\,dy\,dz$

           $+ CL\,dz\,dx + CM\,dz\,dy + CN\,dz\,dz$

           $= (BN - CM)dy\,dz + (CL - AN)dz\,dx + (AM - BL)dx\,dy$

           $= -\alpha\omega$

    (f) $\omega = A\,dx + B\,dy + C\,dz$, $\alpha = L\,dy\,dz + M\,dz\,dx + N\,dx\,dy$

         $\omega\alpha = (AL + BM + CN)dx\,dy\,dz = \alpha\omega$

(g) $\omega = A\,dx + B\,dy + C\,dz$ $\alpha = L\,dx\,dy\,dz$

   $\omega\alpha = 0 = \alpha\omega$

(h) $\omega = A\,dy\,dz + B\,dz\,dx + C\,dx\,dy,$ $\alpha = L\,dx\,dy\,dz$

   $\omega\alpha = 0 = \alpha\omega$

Differentiation of C^1 forms:

A C^1 form is a form in which the coefficients A, B, C are C^1 functions.

(a) <u>0-forms</u>: $\omega = A(x,\,y,\,z)$

   $d\omega = dA = \dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy + \dfrac{\partial A}{\partial z}\,dz$,   a 1-form

(b) <u>1-forms</u>: $\omega = A\,dx + B\,dy + C\,dz$

   $d\omega = (dA)dx + (dB)dy + (dC)dz$

   $= \left(\dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy + \dfrac{\partial A}{\partial z}\,dx\right)dx$

   $+ \left(\dfrac{\partial B}{\partial x}\,dx + \dfrac{\partial B}{\partial y}\,dy + \dfrac{\partial B}{\partial z}\,dz\right)dy$

   $+ \left(\dfrac{\partial C}{\partial x}\,dx + \dfrac{\partial C}{\partial y}\,dy + \dfrac{\partial C}{\partial z}\,dz\right)dz$

   $= \left(\dfrac{\partial C}{\partial y} - \dfrac{\partial B}{\partial z}\right)dy\,dz + \left(\dfrac{\partial A}{\partial z} - \dfrac{\partial C}{\partial x}\right)dz\,dx + \left(\dfrac{\partial B}{\partial x} - \dfrac{\partial A}{\partial y}\right)dx\,dy$,   a 2-form

   $= \begin{vmatrix} dy\,dz & dz\,dx & dx\,dy \\[2mm] \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\[2mm] A & B & C \end{vmatrix}$,   formally (sorry)

(c) <u>2-forms</u>: $\omega = A\,dy\,dz + B\,dz\,dx + C\,dx\,dy$

   $d\omega = (dA)dy\,dz + (dB)dz\,dx + (dC)dx\,dy$

   $= \left(\dfrac{\partial A}{\partial x} + \dfrac{\partial B}{\partial y} + \dfrac{\partial C}{\partial z}\right)dx\,dy\,dz$,   a 3-form

(d) <u>3-forms</u>: $\omega = A\ dx\ dy\ dz$

$$d\omega = (dA)dx\ dy\ dz = 0$$

Notice that if ω is a k-form then $d\omega$ is a (k+1)-form, $k = 0,\ \ldots,\ n-1$.

<u>Exercise</u> :

5.55:   If ω is a C^2 form then

$$d(d\omega) = 0 \quad \text{and} \quad (d\omega)(d\omega) = 0$$

This is true in general. Prove it for 0, 1, 2 and 3-forms in R^3 .

<u>Proposition</u>: If $x = x(u,v)$ and $y = y(u,v)$ are C^1 o-forms then

$$dx\ dy = \frac{\partial (x,y)}{\partial (u,v)}\ du\ dv\ .$$

Hence $dx\ dy = - \dfrac{\partial (y,x)}{\partial (u,v)}\ du\ dv = -\ dy\ dx$.

<u>Proof</u>: $dx = \dfrac{\partial x}{\partial u}\ du + \dfrac{\partial x}{\partial v}\ dv$

$$dy = \frac{\partial y}{\partial u}\ du + \frac{\partial y}{\partial v}\ dv$$

$$dx\ dy = (\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u})\ du\ dv = \frac{\partial (x,y)}{\partial (u,v)}\ du\ dv$$

<u>Corollary</u>: If $A = A(x,y)$ is a C^1 0-form and $x = x(u,v),\ y = y(u,v)$ are C^2 0-forms then

$$\frac{\partial A}{\partial x}\frac{\partial (x,y)}{\partial (u,v)} = \frac{\partial}{\partial u}\ (A\ \frac{\partial y}{\partial v}) - \frac{\partial}{\partial v}\ (A\frac{\partial y}{\partial u})$$

$$\frac{\partial A}{\partial y}\frac{\partial (x,y)}{\partial (u,v)} = \frac{\partial}{\partial v}\ (A\ \frac{\partial x}{\partial u}) - \frac{\partial}{\partial u}\ (A\frac{\partial x}{\partial v})$$

<u>Proof</u>: Consider

$$\omega = A\ dy \qquad\qquad \therefore\quad \omega = A\frac{\partial y}{\partial u}\ du + A\frac{\partial y}{\partial v}\ dv$$

$$\therefore\quad d\omega = \frac{\partial A}{\partial x}\ dx\ dy \qquad\qquad d\omega = [-\frac{\partial}{\partial v}(A\frac{\partial y}{\partial u}) + \frac{\partial}{\partial u}(A\frac{\partial y}{\partial v})]du\ dv$$

$$= \frac{\partial A}{\partial x}\frac{\partial(x,y)}{\partial(u,v)}\ du\ dv$$

Comparing both expressions for $d\omega$ gives the first identity, exchanging

x and y gives the second.

<u>Exercises</u>:

5.56: If $A = A(x,\ y,\ z)$, $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$

Show

$$\frac{\partial A}{\partial z}\frac{\partial(z,x)}{\partial(u,v)} - \frac{\partial A}{\partial y}\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial}{\partial u}(A\frac{\partial x}{\partial v}) - \frac{\partial}{\partial v}(A\frac{\partial x}{\partial u})$$

5.57: (a) If $x = x(u,\ v,\ w)$, $y = y(u,\ v,\ w)$, $z = z(u,\ v,\ w)$ then

$$dy\ dz = \frac{\partial(y,z)}{\partial(v,w)}\ dv\ dw + \frac{\partial(y,z)}{\partial(w,u)}\ dw\ du + \frac{\partial(y,z)}{\partial(u,v)}\ du\ dv$$

$$dx\ dy\ dz = \frac{\partial(x,y,z)}{\partial(u,v,w)}\ du\ dv\ dw$$

(b) If $A = A(x,\ y,\ z)$ then

$$\frac{\partial A}{\partial x}\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{\partial}{\partial u}\{A\frac{\partial(y,z)}{\partial(v,w)}\} + \frac{\partial}{\partial v}\{A\frac{\partial(y,z)}{\partial(w,u)}\} + \frac{\partial}{\partial w}\{A\frac{\partial(y,z)}{\partial(u,v)}\}$$

You may have noticed a strong similarity between addition, multi-
plication and differentiation of forms and the various operations with vectors.

We will discuss this in more detail after we have considered the following important theorem special cases of which are due to Gauss, Green and Stokes; it may be regarded as an extension of the Fundamental Theorem of Calculus.

Theorem 5.3: (Gauss-Green-Stokes)

If S is a piecewise C^2 oriented k-surface with boundary ∂S which is a piecewise C^1 oriented (k-1)-surface and ω is a C^1 (k-1)-form then

$$\omega(\partial S) = d\omega(S)$$

Step I: (Proof for oriented intervals)

$k = 1$ (Fundamental Theorem of Calculus):

Let I be the positively oriented interval $[a,b]$ in R.

$\omega = A(x)$ $\qquad$ $A \in C^1$

$d\omega = A'(x)dx$

$A(b) - A(a) = \int_a^b A'(x)dx$

i.e. $\omega(\partial I) = d\omega(I)$

$k = 2$ (Green's Theorem):

Let I be the positively oriented interval $[a,b] \times [c,d]$

$\omega = A\ dx + B\ dy$ $\qquad$ $A, B \in C^1$

$d\omega = (\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y})\ dx\ dy$
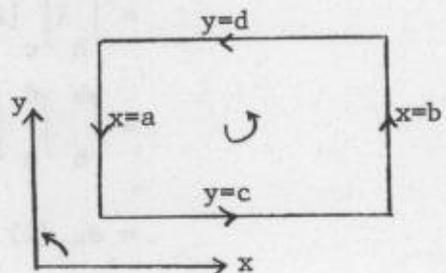
$$\omega(\partial I) = \int_{\partial I} A\,dx + B\,dy$$

$$= \int_a^b A(x,c)dx + \int_c^d B(b,y)dy - \int_a^b A(x,d)dx - \int_c^d B(a,y)dy$$

$$= \int_a^b \{A(x,c) - A(x,d)\}dx + \int_c^d \{B(b,y) - B(a,y)\}dy$$

$$= \int_a^b \{\int_c^d - \frac{\partial A}{\partial y}\,dy\}dx + \int_c^d \{\int_a^b \frac{\partial B}{\partial x}\,dx\}dy$$

$$= \int_I (\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y})\,dx\,dy = d\omega(I)$$

k = 3   (Gauss' Theorem):

Let I be the positively oriented interval $[a,b] \times [c,d] \times [h,k]$

$$\omega = A\,dy\,dz + B\,dz\,dx + C\,dx\,dy, \quad A, B, C \in C^1$$

$$d\omega = (\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z})\,dx\,dy\,dz$$

$$\omega = \omega_1 + \omega_2 + \omega_3$$

$$d\omega = d\omega_1 + d\omega_2 + d\omega_3$$

where $\omega_1 = A\,dy\,dz$, $\omega_2 = B\,dz\,dx$, $\omega_3 = C\,dx\,dy$

$$\omega_1(\partial I) = \int_{\partial I} A\,dy\,dz$$

$$= \int_h^k \{\int_c^d [A(b,y,z) - A(a,y,z)]dy\}dz$$

$$= \int_h^k \int_c^d \int_a^b \frac{\partial A}{\partial x}(x,y,z)dx\,dy\,dz$$

$$= d\omega_1(I)$$
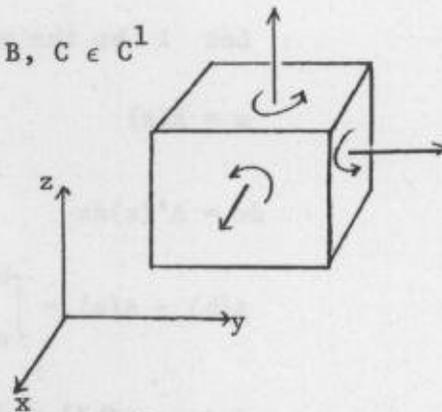
Similarly $\omega_2(\partial I) = d\omega_2(I)$, $\omega_3(\partial I) = d\omega_3(I)$.

$\therefore$ $\omega(\partial I) = d\omega(I)$.

Step II: (Proof for C^2 segments)

We now extend the result to C^2 k-surfaces in R^3 the domain of whose parametrizations are intervals in R^k, k = 1, 2, 3 . The proof for R^2 is obtained by considering z = 0 throughout.

k = 1:

$$\gamma : \begin{cases} x = x(t), \ \gamma \in C^1 [a,b] \\ y = y(t) \\ z = z(t) \end{cases} \qquad S = \gamma[a,b]$$
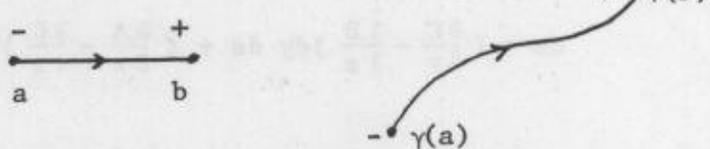
$$\omega = A(x, y, z), \quad A \in C^1$$

$$d\omega = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz$$

$$d\omega(S) = \int_S \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz$$

$$= \int_a^b \{\frac{\partial A}{\partial x} x'(t) + \frac{\partial A}{\partial y} y'(t) + \frac{\partial A}{\partial z} z'(t)\}dt$$

$$= \int_a^b \frac{d}{dt} A(x(t), y(t), z(t))dt$$

$$= A(x(b), y(b), z(b)) - A(x(a), y(a), z(a)), \text{ by Step I (k=1),}$$

$$= A(\gamma(b)) - A(\gamma(a)) = \omega(\partial S) .$$

k = 2 (Stokes' Theorem. Called Green's Theorem in R^2):

$$\sigma : \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \quad \begin{array}{l} \sigma \text{ is } (1\text{-}1), \; C^2(I) \\ I \text{ closed interval in } R^2 \\ S = \sigma(I) \end{array}$$
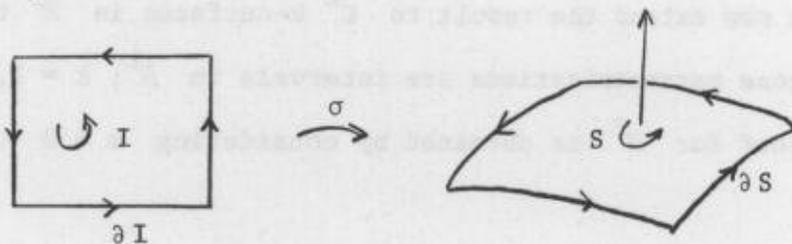
$$\omega = A \, dx + B \, dy + C \, dz \qquad A, B, C \in C^1$$

$$d\omega = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} \right) dy \, dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x} \right) dz \, dx + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \, dy$$

$$\omega = \omega_1 + \omega_2 + \omega_3 \qquad \omega_1 = A \, dx, \quad \omega_2 = B \, dy, \quad \omega_3 = C \, dz$$

$$d\omega = d\omega_1 + d\omega_2 + d\omega_3 \qquad\qquad d\omega_1 = \frac{\partial A}{\partial z} \, dz \, dx - \frac{\partial A}{\partial y} \, dx \, dy, \quad \text{etc.}$$

$$\omega_1(\partial S) = \int_{\partial S} A \, dx = \int_{\partial I} \left(A \, \frac{\partial x}{\partial u} \right) du + \left(A \, \frac{\partial x}{\partial v} \right) dv$$

$$= \int_I \left\{ \frac{\partial}{\partial u} \left(A \, \frac{\partial x}{\partial v} \right) - \frac{\partial}{\partial v} \left(A \, \frac{\partial x}{\partial u} \right) \right\} du \, dv, \quad \text{by Step I } (k = 2)$$

$$= \int_I \left\{ \frac{\partial A}{\partial z} \, \frac{\partial (z,x)}{\partial (u,v)} - \frac{\partial A}{\partial y} \, \frac{\partial (x,y)}{\partial (u,v)} \right\} du \, dv, \quad \text{Exercise 5.56}$$

$$= \int_S \left\{ \frac{\partial A}{\partial z} \, dz \, dx - \frac{\partial A}{\partial y} \, dx \, dy \right\}$$

$$= d\omega_1(S)$$

Similarly $\omega_2(\partial S) = d\omega_2(S)$, $\omega_3(\partial S) = d\omega_3(S)$.

$$\therefore \quad \omega(\partial S) = d\omega(S) .$$

$k = 3$ (Gauss' Theorem):

$$\nu : \begin{cases} x = x(u,v,w) & \nu \text{ is } (1\text{-}1), \ C^2(I) \\ y = y(u,v,w) & I \text{ closed interval in } R^3 \\ z = z(u,v,w) & S = \nu(I) \end{cases}$$
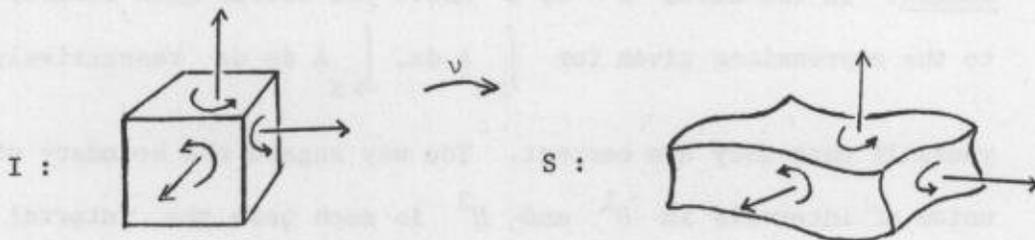
$$\omega = A \ dy \ dz + B \ dz \ dx + C \ dx \ dy, \quad A, B. \ C \in C^1$$

$$d\omega = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) dx \ dy \ dz$$

$$\omega = \omega_1 + \omega_2 + \omega_3 \qquad\qquad \omega_1 = A \ dy \ dz , \quad \text{etc.}$$

$$d\omega = d\omega_1 + d\omega_2 + d\omega_3 \qquad\qquad d\omega_1 = \frac{\partial A}{\partial x} dx \ dy \ dz , \quad \text{etc.}$$

$$\omega_1(\partial S) = \int_{\partial S} A \ dy \ dz$$

$$= \int_{\partial I} A \left\{ \frac{\partial (y,z)}{\partial (v,w)} dv \ dw + \frac{\partial (y,z)}{\partial (w,u)} dw \ du + \frac{\partial (y,z)}{\partial (u,v)} du \ dv \right\}, \qquad \text{Exercise 5.57 (a)}$$

$$= \int_I \left\{ \frac{\partial}{\partial u} \left[A \frac{\partial (y,z)}{\partial (v,w)} \right] + \frac{\partial}{\partial v} \left[A \frac{\partial (y,z)}{\partial (w,u)} \right] + \frac{\partial}{\partial w} \left[A \frac{\partial (y,z)}{\partial (u,v)} \right] \right\} du \ dv \ dw , \quad \text{Step I(k=3)}$$

$$= \int_I \frac{\partial A}{\partial x} \frac{\partial (x,y,z)}{\partial (u,v,w)} \, du \, dv \, dw \, , \qquad \text{Exercise 5.57} \qquad \text{(b)}$$

$$= \int_S \frac{\partial A}{\partial x} \, dx \, dy \, dz = d\omega_1(S)$$

Similarly $\quad \omega_2(\partial S) = d\omega_2(S) \, , \quad \omega_3(\partial S) = d\omega_3(S) \, .$

$$\therefore \quad \omega(\partial S) = d\omega(S) \, .$$

Remark: In the cases $k = 2, 3$ above you should give careful consideration to the expressions given for $\int_{\partial S} A \, dx, \int_{\partial S} A \, dy \, dz$ respectively and convince yourself that they are correct. You may regard the boundary of I as the union of intervals in R^1 and R^2 in each case the interval being the domain of a parametrization of a segment of ∂S.

Step III: (Piecewise C^2 k-surfaces)

We have proved the Gauss-Green-Stokes Theorem for surfaces which are smooth (1-1) images of intervals. The theorem also holds for any object S which is a finite union of such images - the common parts of their boundaries having opposite orientations so that the contributions to $\omega(\partial S)$ arising from these cancel each other. e.g.

(1)
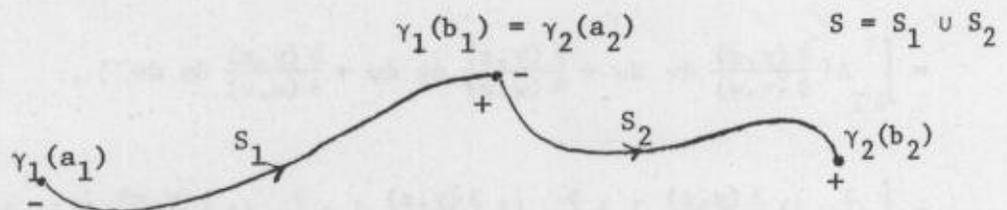
$$\omega(\partial S) = \omega(\gamma_2(b_2)) - \omega(\gamma_1(a_1))$$

$$= \omega(\gamma_2(b_2)) - \omega(\gamma_2(a_2)) + \omega(\gamma_1(b_1) - \omega(\gamma_1(a_1))$$

$$= \omega(\partial S_2) + \omega(\partial S_1)$$

$$= d\omega(S_2) + d\omega(S_1) = d\omega(S)$$

(ii)
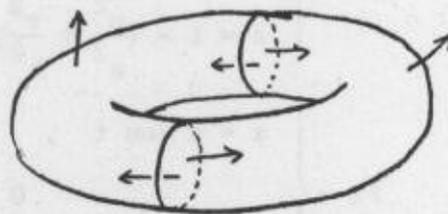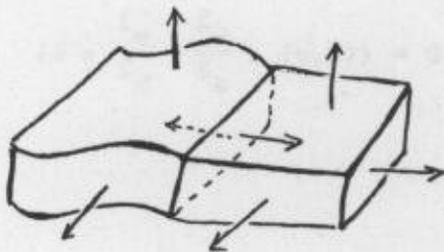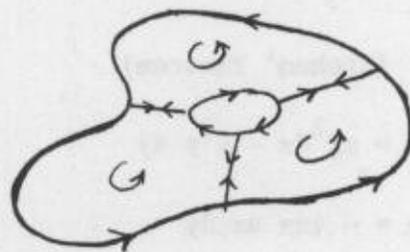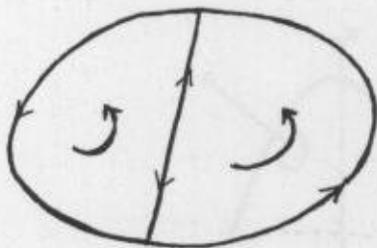
Example 1: (Green's Theorem)

$$\omega = x^2 y \, dx - y^2 x \, dy$$

$$d\omega = - (x^2 + y^2) dx \, dy$$
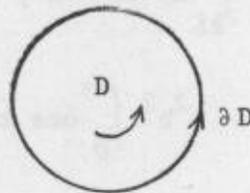
$$D : \begin{cases} x = r \cos \theta , & 0 \le r \le 1 \\ \\ y = r \sin \theta , & 0 \le \theta \le 2\pi \end{cases} , \quad \partial D : \begin{cases} x = \cos \theta , \\ & 0 \le \theta \le 2\pi \\ y = \sin \theta , \end{cases}$$

$$\omega(\partial D) = \int_{\partial D} x^2 y \ dx - y^2 x \ dy = -2 \int_0^{2\pi} \cos^2\theta \ \sin^2\theta \ d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} \sin^2 2\theta \ d\theta = -\frac{1}{4} \int_0^{2\pi} (1 - \cos 4\theta) d\theta = -\frac{\pi}{2}$$

$$d\omega(D) = -\int_{-D} (x^2 + y^2) dx \ dy = -\int_0^{2\pi}\int_0^1 r^3 \ dr \ d\theta = -\frac{\pi}{2}$$

$$\therefore \ \omega(\partial D) = d\omega(D) \ .$$

Example 2: (Stokes' Theorem)

$$\omega = xy^2 dx - x^2 y \ dy$$

$$d\omega = -4xy \ dx \ dy$$

$$\Sigma : \begin{cases} x = u \\ y = v \\ z = 1 - (\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2}) \ , \quad (u,v) \in D = \{(u,v) : \dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} \le 1\} \end{cases}$$

$$\partial\Sigma : \begin{cases} x = a \cos t \ , \\ \qquad\qquad\qquad 0 \le t \le 2\pi \ . \\ y = b \sin t \ , \end{cases}$$
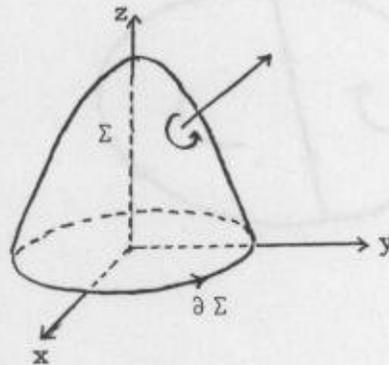
$$\omega(\partial\Sigma) = \int_{\partial\Sigma} xy^2 dx - x^2 y \ dy = -\int_0^{2\pi} \{a^2 b^2 \cos t \sin^3 t + a^2 b^2 \cos^3 t \sin t\} dt$$

$$= -a^2 b^2 \int_0^{2\pi} \cos t \sin t \ dt = 0$$

$$d\omega(\Sigma) = -\int_{\Sigma} 4xy \ dx \ dy = -\int_D 4uv \ du \ dv = 0$$

$$\therefore \ \omega(\partial\Sigma) = d\omega(\Sigma) \ .$$

You could also use, for example, the parameters (r, θ) instead of (u,v) where $x = ar \cos \theta$, $y = br \sin \theta$, $z = 1-r^2$ $0 \le r \le 1$, $0 \le \theta \le 2\pi$.

Example 3: (Gauss' Theorem)

$$\omega = (x + y^2) dy \, dz$$

$$d\omega = dx \, dy \, dz$$

$$K : \begin{cases} x = r \sin \phi \ \cos \theta , & 0 \le r \le 1 , \\ y = r \sin \phi \ \sin \theta , & 0 \le \phi \le \pi , \\ z = r \cos \phi & 0 \le \theta \le 2\pi. \end{cases}$$

$$\partial K : \begin{cases} x = \sin \phi \ \cos \theta , & 0 \le \phi \le \pi , \\ y = \sin \phi \ \cos \theta , & 0 \le \theta \le 2\pi. \\ z = \cos \phi \end{cases} \qquad D = \{(\phi, \theta) : 0 \le \phi \le \pi, \ 0 \le \theta \le 2\pi\}$$
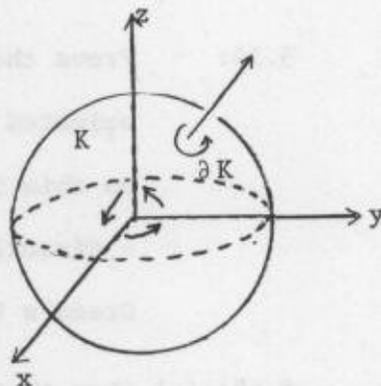
$$d\omega(K) = \frac{4}{3} \pi$$

$$\omega(\partial K) = \int_{\partial K} (x + y^2) dy \, dz = \int_D (x + y^2) \frac{\partial (y,z)}{\partial (\phi, \theta)} d\phi \, d\theta \quad *$$

$$= \int_0^{2\pi} \{ \int_0^\pi (\sin \phi \cos \theta + \sin^2 \phi \sin^2 \theta) \sin^2 \phi \cos \phi \ d\phi \ d\theta$$

$$= \int_0^{2\pi} \{ \int_0^\pi \sin^3 \phi \cos^2 \theta \ d\phi \} d\theta = \frac{4}{3} \int_0^{2\pi} \cos^2 \theta \ d\theta = \frac{4}{3} \pi$$

* 'dϕ dθ' is consistent with the positive orientation of the unit ball; 'dθ dϕ' would give negative orientation.
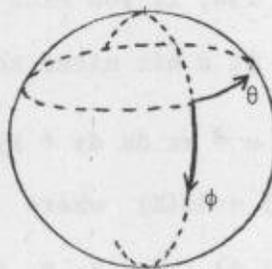
Exercises:

5.58: Let D be the hemispherical surface $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

Verify that $\omega(\partial D) = d\omega(D)$ for the form $\omega = xy\, dx + x^2 dy$.

5.59: Prove that $\int_\gamma (x^2 + 2y)dx + (y - x)dy = -13$ where γ is the oriented curve indicated.

Do this in two ways

– directly and by

Green's Theorem.
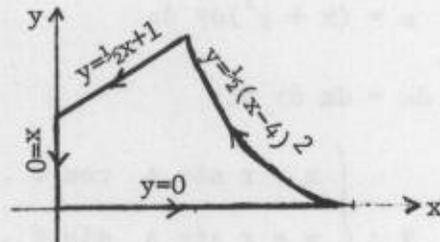
5.60: (a) Show that the content of a region D in R^n to which the GGS Theorem applies is given by

$$\mu_n(D) = \omega(\partial D)$$

where: in R^1, $\omega = x$; in R^2, $\omega = \frac{1}{2}(x\, dy - y\, dx)$;

in R^3, $\omega = \frac{1}{3}(x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$.

(b) Use (a) to find the area of the region bounded by the ellipse

$x = a \cos\theta$, $y = b \sin\theta$, $0 \leq \theta \leq 2\pi$.

(c) Find the area of the region bounded by the curve $x^{2/3} + y^{2/3} = 1$.

(d) Would any other forms ω suffice in part (a)?

5.61: You were asked in Exercise 3.22, page 133, to evaluate an integral in two ways. Now do it by at least two more. Do Exercise 3.25, page 134; if you didn't have the courage to do it before you may find it a bit nicer now.

5.62: Let $\omega = xz\, dx\, dy + xy\, dy\, dz + yz\, dz\, dx$; verify Gauss' Theorem $\omega(\partial K) = d\omega(K)$ where K is the positively oriented tetrahedron $\{(x,y,z) : 0 \leq x, y, z, 1 \geq x + y + z\}$.

5.63: Show that $\int_\gamma (x^2 + 3y)dx + (y - 2x)dy = - 25$ where γ is the oriented curve indicated in the diagram.

Do this in two ways

- directly and by

Green's Theorem .
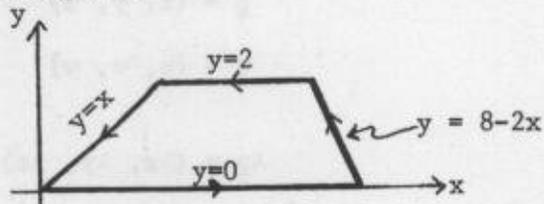
5.64: Find a 1-form ω for which $d\omega = (x^2 + y^2)dx\,dy$ and use this to evaluate $\int_D (x^2 + y^2)dx\,dy$ where D is the region inside the square $|x| + |y| = 4$ and outside the circle $x^2 + y^2 = 1$.

5.65: Let ω and α be k- and ℓ-forms respectively which are of class C^1 . Prove that if S is an oriented $k + \ell + 1$-surface to which the Gauss-Green-Stokes Theorem is applicable then

$$(-1)^k \omega \, d \, \alpha(S) = \omega \, \alpha(\partial S) - (d\omega) \, \alpha(S)$$

Where have you seen this before?

5.66: Prove Green's Theorem for a disc in R^2 in two ways: first by change of variable assuming Green's Theorem for an interval and taking the limit as $\epsilon \to 0$ (diagram on the left) and second by evaluating $\int \frac{\partial A}{\partial x} dx\,dy$ (or $\int \frac{\partial A}{\partial y} dx\,dy$) directly by Fubini's Theorem.
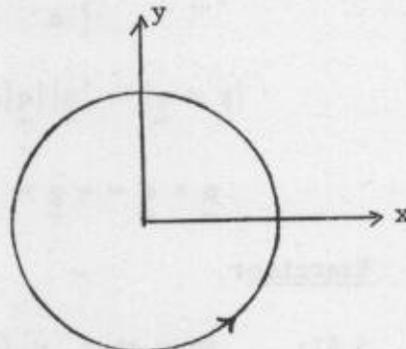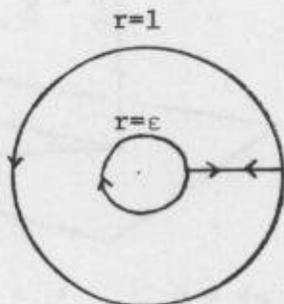
Vectors and vector operations in R^3 :

$$\underset{\sim}{p} = (x, y, z)$$

$$\underset{\sim}{q} = (u, v, w)$$

vectors in R^3, $\lambda \in R$ scalar

$$\lambda\underset{\sim}{p} = (\lambda x, \lambda y, \lambda z)$$

$$\underset{\sim}{p} + \underset{\sim}{q} = (x + u, y + v, z + w)$$

If $\underset{\sim}{i} = (1, 0, 0)$, $\underset{\sim}{j} = (0, 1, 0)$, $\underset{\sim}{k} = (0, 0, 1)$ then

$$\underset{\sim}{p} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} \qquad \underset{\sim}{q} = u\underset{\sim}{i} + v\underset{\sim}{j} + w\underset{\sim}{k}$$

Inner product (dot product):

$$\underset{\sim}{p} \cdot \underset{\sim}{q} \overset{def}{=} xu + yv + zw$$

$$= |\underset{\sim}{p}||\underset{\sim}{q}| \cos \theta$$

where $|\underset{\sim}{p}| = \sqrt{\underset{\sim}{p} \cdot \underset{\sim}{p}}$, $|\underset{\sim}{q}| = \sqrt{\underset{\sim}{q} \cdot \underset{\sim}{q}}$

CBS : $\underset{\sim}{p} \cdot \underset{\sim}{q} \leq |\underset{\sim}{p}||\underset{\sim}{q}|$
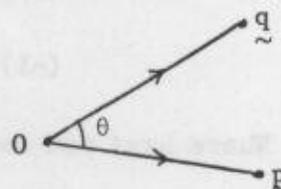
Vector product (cross product):

$$\underset{\sim}{p} \times \underset{\sim}{q} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ x & y & z \\ u & v & w \end{vmatrix} = (yw - zv)\underset{\sim}{i} + (zu - xw)\underset{\sim}{j} + (xv - yu)\underset{\sim}{k}$$

$$|\underset{\sim}{p} \times \underset{\sim}{q}| = |\underset{\sim}{p}||\underset{\sim}{q}| \sin \theta$$

$$\underset{\sim}{p} \times \underset{\sim}{q} = - \underset{\sim}{q} \times \underset{\sim}{p}$$
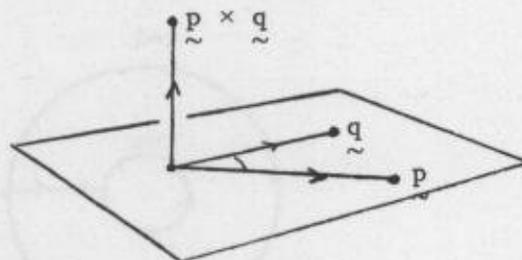
Exercise:

5.67: Show that $\underset{\sim}{p} \cdot (\underset{\sim}{p} \times \underset{\sim}{q}) = \underset{\sim}{q} \cdot (\underset{\sim}{p} \times \underset{\sim}{q}) = 0$.

Differential operations:

Definition: A <u>scalar</u> <u>field</u> on a set D is a function $f : D \to R$.

Gradient:

$$\underset{\sim}{\nabla} \overset{def}{=} \underset{\sim}{i} \frac{\partial}{\partial x} + \underset{\sim}{j} \frac{\partial}{\partial y} + \underset{\sim}{k} \frac{\partial}{\partial z}$$

$$f : R^3 \to R, \quad f = f(x, y, z), \quad f \in C^1$$

$$\text{grad } f \overset{def}{=} \nabla f = \underset{\sim}{i} \frac{\partial f}{\partial x} + \underset{\sim}{j} \frac{\partial f}{\partial y} + \underset{\sim}{k} \frac{\partial f}{\partial z}$$

Recall: (a) The directional derivative of f in the direction $\underset{\sim}{q} = (u,v,w)$ is

$$[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}] \begin{bmatrix} u \\ v \\ w \end{bmatrix} = u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z} = \underset{\sim}{\nabla} f \cdot \underset{\sim}{q}$$

(b) If $|q| = 1$, $\underset{\sim}{\nabla} f \cdot \underset{\sim}{q}$ is maximized by $\underset{\sim}{q} = \underset{\sim}{\nabla} f / |\underset{\sim}{\nabla} f|$

$$= \frac{1}{\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + (\frac{\partial f}{\partial z})^2}} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \quad \text{i.e. } \underset{\sim}{\nabla} f \text{ points in the}$$

direction of maximum rate of change of f (Exercise 4.22 p. 164).

(c) The vector $\underset{\sim}{\nabla} f(x, y, z)$ is normal to the level surface

$$f(x, y, z) = c \quad \text{for each } (x, y, z) \text{ on the surface if } |\underset{\sim}{\nabla} f| \neq 0.$$

Definition: A <u>vector</u> <u>field</u> on a set D is a function $\underset{\sim}{V} : D \to R^3$

i.e. $\underset{\sim}{V}(x, y, z) = (A(x, y, z), B(x, y, z), C(x, y, z)) = A\underset{\sim}{i} + B\underset{\sim}{j} + C\underset{\sim}{k}$

For example:

(a) A mass distribution in space exerts a force $\underset{\sim}{F}(x, y, z)$ on a unit point mass at any point (x, y, z). $\underset{\sim}{F}$ is a vector field on R^3 (gravitational field).

(b) A system of differential equations

$$\frac{dx}{dt} = f(x, y, z)$$

$$\frac{dy}{dt} = g(x, y, z)$$

$$\frac{dz}{dt} = h(x, y, z)$$

$\underset{\sim}{v} = (f, g, h)$

(x, y, z)
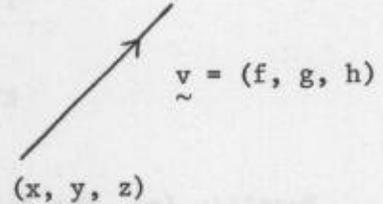
This can be regarded as a velocity field or a tangent field. Solving the system of equations with initial condition $(x(0), y(0), z(0)) = (x_o, y_o, z_o)$ means finding a curve $\gamma(t) = (x(t), y(t), z(t))$ with $\gamma(0) = (x_o, y_o, z_o)$ such that the tangent direction at any point $\gamma(t)$ is $(f(\gamma(t)), g(\gamma(t)), h(\gamma(t)))$.

(c) If $f = f(x, y, z)$ is a scalar field then

$$\underset{\sim}{\nabla} f = \left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k \right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \text{ is a vector field}$$

(called a gradient field).

Addition, scalar multiplication, dot and cross products of fields may be taken pointwise. We have considered the gradient of scalar fields; there are also differential operations defined on vector fields.

<u>Divergence</u> <u>and</u> <u>curl</u> <u>of</u> <u>a</u> <u>vector</u> <u>field</u>:

$$\underset{\sim}{\nabla} = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z} , \quad \underset{\sim}{V} = A\underset{\sim}{i} + B\underset{\sim}{j} + C\underset{\sim}{k}$$

$$\text{div } V \overset{\text{def}}{=} \underset{\sim}{\nabla}\cdot\underset{\sim}{V} = \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z} \right)\cdot(A\underset{\sim}{i} + B\underset{\sim}{j} + C\underset{\sim}{k})$$

$$= \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}$$

$$\text{Curl } \underset{\sim}{V} \overset{\text{def}}{=} \underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A & B & C \end{vmatrix}$$

$$= \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} \right)\underset{\sim}{i} + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x} \right)\underset{\sim}{j} + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right)\underset{\sim}{k}$$

All of the algebraic and differential operations on vector fields are special cases of the operations we have defined on forms.

<u>Vectors</u> <u>and</u> <u>forms</u>:

$$A(x, y, z)$$

$$A(x, y, z)$$

Scalar field

$$A(x, y, z)dx\ dy\ dz$$

$$\nu = A\ dx + B\ dy + C\ dz$$

Vector field

$$\underset{\sim}{V} = A\underset{\sim}{i} + B\underset{\sim}{j} + C\underset{\sim}{k}$$

$$\nu^* = A\ dy\ dz + B\ dz\ dx + C\ dx\ dy$$
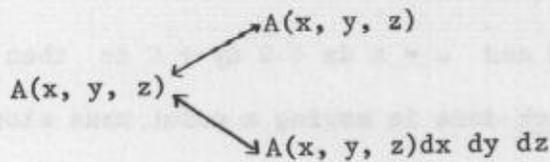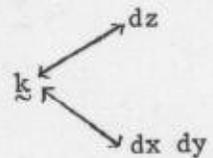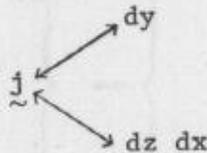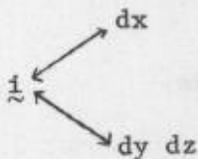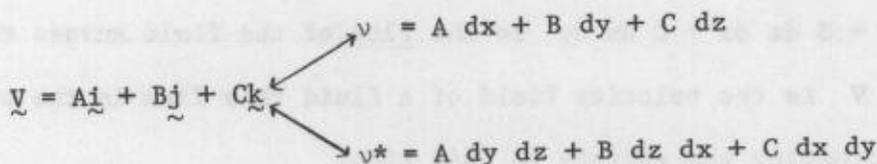
$$\underset{\sim}{W} = P\underset{\sim}{i} + Q\underset{\sim}{j} + R\underset{\sim}{k} \quad \nearrow \quad \omega = P\,dx + Q\,dy + R\,dz$$

$$\searrow \quad \omega* = P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$

$$A\underset{\sim}{W} = A\,P\underset{\sim}{i} + A\,Q\underset{\sim}{j} + A\,R\underset{\sim}{k} \quad \nearrow \quad A\omega = A\,P\,dx + A\,Q\,dy + A\,R\,dz$$

$$\searrow \quad A\omega* = A\,P\,dy\,dz + A\,Q\,dz\,dx + A\,R\,dx\,dy$$

$$\underset{\sim}{V}\cdot\underset{\sim}{W} = AP + BQ + CR \leftrightarrow \nu\omega* = (AP + BQ + CR)dx\,dy\,dz$$

$$V \times W = (BR - CQ)\underset{\sim}{i} \left.\begin{array}{l} \\ + (CP - AR)\underset{\sim}{j} \\ + (AQ - BP)\underset{\sim}{k} \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} \nu\omega = (BR - CQ)dy\,dz \\ + (CP - AR)dz\,dx \\ + (AQ - BP)dx\,dy \end{array}\right.$$

$$\underset{\sim}{\nabla} A = \frac{\partial A}{\partial x}\underset{\sim}{i} + \frac{\partial A}{\partial y}\underset{\sim}{j} + \frac{\partial A}{\partial z}\underset{\sim}{k} \leftrightarrow d\,A = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{V} = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \leftrightarrow d\nu* = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right)dx\,dy\,dz$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right)\underset{\sim}{i} \left.\begin{array}{l} \\ + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right)\underset{\sim}{j} \\ + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)\underset{\sim}{k} \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} d\nu = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right)dy\,dz \\ + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right)dz\,dx \\ + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)dx\,dy \end{array}\right.$$

If $\underset{\sim}{V} = A\underset{\sim}{i} + B\underset{\sim}{j} + C\underset{\sim}{k}$ is a force field and $\omega = A\,dx + B\,dy + C\,dz$ then $\omega(\gamma) = \int_\gamma A\,dx + B\,dy + C\,dz$ is the <u>work</u> done in moving a point mass along the curve γ. If $\omega* = A\,dy\,dz + B\,dz\,dx + C\,dx\,dy$ then $\omega*(S) = \int_S A\,dy\,dz + B\,dz\,dx + C\,dx\,dy$ is the <u>flux</u> of the field across the surface S. When V is the velocity field of a fluid this flux is the net volume of liquid crossing the surface in unit time.

Vector Formulations of Green's, Stokes' and Gauss' Theorems:

<u>Green's Theorem</u>: D is a 'nice' oriented region in R^2 .
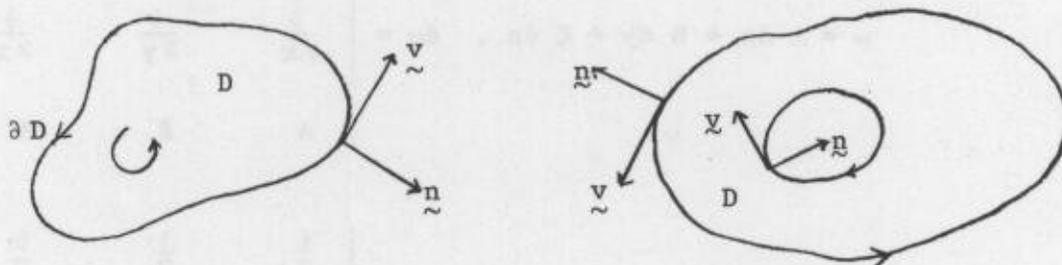
$$\int_D \underset{\sim}{\nabla} \cdot \underset{\sim}{V} \, d\mu_2 \; = \; \int_{\partial D} \underset{\sim}{V} \cdot \underset{\sim}{n}_1 \, d\ell \quad \text{(Green's Theorem)}$$

Explanation:

$$\int_D \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} \right) dx \, dy = \int_{\partial D} (A \, dy - B \, dx)$$

i.e. $d\omega(D) = \omega(\partial D)$ where $\omega = A \, dy - B \, dx$.

$$\underset{\sim}{V} = A\underset{\sim}{i} + B\underset{\sim}{j} \quad , \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{V} = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}$$

$$\underset{\sim}{v}(t) = x'(t)\underset{\sim}{i} + y'(t)\underset{\sim}{j} \quad , \quad \underset{\sim}{n}(t) = y'(t)\underset{\sim}{i} - x'(t)\underset{\sim}{j}$$

$$\underset{\sim}{n}_1(t) = \underset{\sim}{n}(t)/|\underset{\sim}{n}(t)| \quad , \quad d\ell = |\underset{\sim}{v}|dt = |\underset{\sim}{n}|dt$$

<u>Stokes' Theorem</u>: S is a 'nice' oriented surface in R^3 .

$$\int_S \left(\underset{\sim}{\nabla} \times \underset{\sim}{V} \right) \cdot \underset{\sim}{n}_1 \, da = \int_{\partial S} \underset{\sim}{V} \cdot \underset{\sim}{v}_1 \, d\ell \quad \text{(Stokes' Theorem)}$$

Explanation: $d\omega(S) = \omega(\partial S)$,

$$\omega = A\,dx + B\,dy + C\,dz \ , \quad d\omega = \begin{vmatrix} dy\,dz & dz\,dx & dx\,dy \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A & B & C \end{vmatrix}$$

$$\underset{\sim}{V} = A\underset{\sim}{i} + B\underset{\sim}{j} + C\underset{\sim}{k} \ , \quad \underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A & B & C \end{vmatrix}$$

$$\underset{\sim}{n} = \frac{\partial(y,z)}{\partial(u,v)}\,\underset{\sim}{i} + \frac{\partial(z,x)}{\partial(u,v)}\,\underset{\sim}{j} + \frac{\partial(x,y)}{\partial(u,v)}\,\underset{\sim}{k} \ , \quad \underset{\sim}{n}_1 = \underset{\sim}{n}/|\underset{\sim}{n}|$$

$$da = |\underset{\sim}{n}|\ du\,dv$$

Gauss' Theorem: K is a 'nice' oriented solid in R^3 .
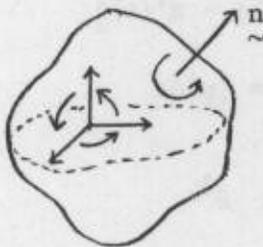
$$\int_K \underset{\sim}{\nabla}\cdot\underset{\sim}{V}\,d\mu_3 = \int_{\partial K} \underset{\sim}{V}\cdot\underset{\sim}{n}_1\,da \quad \text{(Gauss' Theorem)}$$

Explanation: $d\omega(K) = \omega(\partial K)$

$$\omega = A\,dy\,dz + B\,dz\,dx + C\,dx\,dy \ , \quad d\omega = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right)dx\,dy\,dz$$

$$\underset{\sim}{V} = A\underset{\sim}{i} + B\underset{\sim}{j} + C\underset{\sim}{k} \ , \quad \underset{\sim}{\nabla}\cdot\underset{\sim}{V} = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}$$

<u>Corollary 5.3.1</u>: (Green's Identities)

If D is a region in R^3 to which Gauss' Theorem is applicable and ϕ , ψ are of class C^2 on D then

$$\int_D \{\nabla\psi\cdot\nabla\phi + \psi\nabla^2\phi\}d\mu_3 = \int_{\partial D} \psi\,\frac{\partial\phi}{\partial n_1}\,da$$

(Green's first identity)

$$\int_D \{\psi\nabla^2\phi - \phi\nabla^2\psi\}d\mu_3 = \int_{\partial D} \{\psi\,\frac{\partial\phi}{\partial n_1} - \phi\,\frac{\partial\psi}{\partial n_1}\}da$$

(Green's second identity)

where $\nabla^2\phi \overset{\text{def}}{=} \nabla\cdot\nabla\phi = \text{div}(\text{grad }\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$; ∇^2 is called the <u>Laplacian</u> and $\frac{\partial\phi}{\partial n_1} \overset{\text{def}}{=} \nabla\phi\cdot n_1$ (n_1 : unit normal to ∂D).

The identities also hold in R^2 for a region D to which Green's Theorem is applicable with $d\mu_3$, da replaced by $d\mu_2$, $d\ell$ respectively.

<u>Proof</u>: In Gauss' Theorem let $V = \psi \nabla\phi$

then $V\cdot n_1 = \psi\nabla\phi\cdot n_1 = \psi\,\frac{\partial\phi}{\partial n_1}$

$$\nabla\cdot(\psi\nabla\phi) = \nabla\psi\cdot\nabla\phi + \psi\nabla^2\phi$$

so that the first identity follows. To obtain the second identity rewrite the first with ψ and ϕ interchanged; subtract from the first identity.

<u>Applications</u>:

The physical significance of div and curl: Let V be the velocity field of a steady flow of fluid in space. Let K be a small closed ball in the fluid. Gauss' Theorem states

$$\int_K \underset{\sim}{\nabla} \cdot \underset{\sim}{V} d\mu_3 = \int_{\partial K} \underset{\sim}{V} \cdot \underset{\sim}{n}_1 \, da$$

The surface integral gives the net flow of fluid out of the ball per unit time which is positive or negative depending on whether sources or sinks predominate in the ball. Thus $\underset{\sim}{\nabla} \cdot \underset{\sim}{V}$ measures the 'density of sources' (cf. Exercise 5.72). A field $\underset{\sim}{V}$ is said to be '<u>source-free</u>' if $\underset{\sim}{\nabla} \cdot \underset{\sim}{V} = 0$.

Taking a small plane surface element S with normal $\underset{\sim}{n}$ in the fluid Stokes' Theorem gives

$$\int_S (\underset{\sim}{\nabla} \times \underset{\sim}{V}) \cdot \underset{\sim}{n}_1 da = \int_{\partial S} \underset{\sim}{V} \cdot \underset{\sim}{v}_1 \, d\ell$$

The integral on the right measures the 'circulation' of the fluid around ∂S (i.e. around the direction $\underset{\sim}{n}$). $\underset{\sim}{\nabla} \times \underset{\sim}{V}$ is the circulation density of the flow i.e. its component in any direction measures the 'rotation' of the field about that direction (cf. Exercise 5.72). If $\underset{\sim}{\nabla} \times \underset{\sim}{V} = 0$ the flow is called '<u>irrotational</u>' . A good discussion of the physical implications of these theorems may be found in Courant 'Differential and Integral Calculus' Vol II, Ch. V and Buck, Ch. 7.

<u>Lemma 5.3.1</u>: Let I be a closed interval in R^n and f a continuous non-negative function on I.   Then

$$\int_I f = 0 \Rightarrow f(p) = 0 , \, \forall \, p \in I .$$

<u>Proof</u>: <u>Exercise 5.68</u>: What can be said if we replace 'continuous' by 'integrable'?

<u>Corollary 5.3.2</u>: Let D be an open subset of R^3 (R^2) such that Gauss' (Green's) Theorem is applicable to $\overline{D}$. Show that if $h \in C(D)$ and $\theta : \partial D \to R$ then there is at most one function $\phi \in C^2(D)$ such that

$$\nabla^2 \phi = h , \quad \text{on } D \quad \text{(\underline{Poisson's} \underline{Equation})}$$

and $\qquad \phi(p) = \theta(p) , \quad p \in \partial D$

<u>Proof</u>: Suppose ϕ_1 and ϕ_2 are two such functions.

Let $\psi = \phi_1 - \phi_2$, $\nabla^2 \psi = \nabla^2 \phi_1 - \nabla^2 \phi_2 = h - h = 0$

and $\psi(p) = \phi_1(p) - \phi_2(p) = \theta(p) - \theta(p) = 0$, $p \in \partial D$

$\therefore \displaystyle\int_D |\nabla \psi|^2 + \int_D \psi \nabla^2 \psi = \int_{\partial D} \psi \frac{\partial \psi}{\partial n_1}$ (from Green's first identity)

$$= 0 \ (\psi(p)) = 0 , \quad \text{if } p \in \partial D)$$

$\therefore \displaystyle\int_D |\nabla \psi|^2 = 0$ (since $\nabla^2 \psi = 0$) $\Rightarrow \nabla \psi = 0$ (Lemma 5.3.1)$\Rightarrow \psi = $ constant

$$\Rightarrow \psi = 0 \quad (\psi(p) = 0 , \text{ if } p \in \partial D)$$

$$\Rightarrow \phi_1 = \phi_2 \quad (\psi = \phi_1 - \phi_2) .$$

<u>Exercises</u>:

5.69: Show that a solution to <u>Laplace's</u> <u>Equation</u> $\nabla^2 \phi = 0$ cannot have a strict interior relative extremum [suppose $M = \phi(p_o) > \phi(p)$ for all $p \neq p_o$ in a neighbourhood U of p_o. Let $D = \{p : \phi(p) > M - \varepsilon\}$; find two solutions to $\nabla^2 \phi = 0$ such that $\phi(p) = M - \varepsilon$, $p \in \partial D$ contradicting the preceding Example. Proofs of this result not depending on Gauss' Theorem may be found in Buck page 358 (Theorem 14) and Bartle page 271 (Exercise 21V).

5.70:　　Let $\theta : \begin{cases} x = x(u,v) \\ \\ y = y(u,v) \end{cases}$ define a coordinate system (u,v) . The

system is __orthogonal__ if $\theta'^T \theta' = \text{diag}[h_1^2, h_2^2]$

i.e.
$$
\begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\[2em] \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{bmatrix}
\begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2em] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}
= \begin{bmatrix} h_1^2 & 0 \\[2em] 0 & h_2^2 \end{bmatrix}
$$

(a)　Show that

$$\frac{\partial (x,y)}{\partial (u,v)} = h_1 h_2$$

$$\frac{\partial u}{\partial x} = \frac{1}{h_1^2} \frac{\partial x}{\partial u} \ , \ \frac{\partial u}{\partial y} = \frac{1}{h_1^2} \frac{\partial y}{\partial u} \ , \ \frac{\partial v}{\partial x} = \frac{1}{h_2^2} \frac{\partial x}{\partial v} \ , \ \frac{\partial v}{\partial y} = \frac{1}{h_2^2} \frac{\partial y}{\partial v}$$

$$\frac{\partial \phi}{\partial x} \, dy - \frac{\partial \phi}{\partial y} \, dx = \frac{h_2}{h_1} \frac{\partial \phi}{\partial u} \, dv - \frac{h_1}{h_2} \frac{\partial \phi}{\partial v} \, du$$

(b)
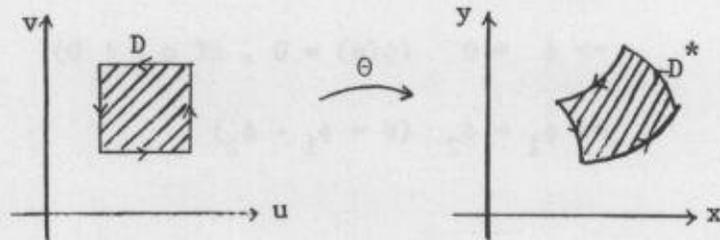
If $\nabla^2 \phi = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}$, $\displaystyle\int_{D*} \nabla^2 \phi \ dx \ dy = \int_{\partial D*} \left(\frac{\partial \phi}{\partial x} \, dy - \frac{\partial \phi}{\partial y} \, dx \right)$

$$\therefore \int_D \nabla^2 \phi \ h_1 h_2 \ du \ dv = \int_{\partial D} \left(\frac{h_2}{h_1} \frac{\partial \phi}{\partial u} \, dv - \frac{h_1}{h_2} \frac{\partial \phi}{\partial v} \, du \right)$$

$$D = [u_o, u_o + \varepsilon] \times [v_o, v_o + \eta]$$

$$= \int_{u_o}^{u_o+\varepsilon} [\frac{h_1}{h_2}\frac{\partial \phi}{\partial v}(u,v_o+\eta) - \frac{h_1}{h_2}\frac{\partial \phi}{\partial v}(u,v_o)]du + \int_{v_o}^{v_o+\eta} [\frac{h_2}{h_1}\frac{\partial \phi}{\partial u}(u_o+\varepsilon,v) - \frac{h_2}{h_1}\frac{\partial \phi}{\partial u}(u_o v)]dv$$

Divide this equation by $\varepsilon \eta$ and, using the Mean Value Theorems for derivatives and integrals, show

$$\nabla^2\phi = \frac{1}{h_1 h_2} [\frac{\partial}{\partial u} \{\frac{h_2}{h_1}\frac{\partial \phi}{\partial u}\} + \frac{\partial}{\partial v} \{\frac{h_1}{h_2}\frac{\partial \phi}{\partial v}\}]$$

(c) If $x = r \cos \theta$, $y = r \sin \theta$ then $h_1 = 1$, $h_2 = r$ and

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r} (r \frac{\partial \phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

(Compare Exercise 4.42 p. 195)

5.71: (a) If
$$\Theta : \begin{cases} x = x(u, v, w) \\ y = y(u, v, w) \\ z = z(u, v, w) \end{cases}$$
satisfies $\Theta'^T\Theta' = \text{diag}[h_1^2, h_2^2, h_3^2]$

show that

$$\nabla^2\phi = \frac{1}{h_1 h_2 h_3} [\frac{\partial}{\partial u} \{\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial u}\} + \frac{\partial}{\partial v} \{\frac{h_3 h_1}{h_2}\frac{\partial \phi}{\partial v}\} + \frac{\partial}{\partial w} \{\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial w}\}]$$

where $\nabla^2\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$.

(b) If $x = r \sin \phi \cos \theta$, $y = r \sin \phi \sin \theta$, $z = r \cos \phi$

$((r, \phi, \theta)$ are spherical polar coordinates) show

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r} (r^2 \frac{\partial \phi}{\partial r}) + \frac{1}{r^2 \sin \phi}\frac{\partial}{\partial \phi} (\sin \phi \frac{\partial \phi}{\partial \phi}) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 \phi}{\partial \theta^2}$$

$[h_1 = 1, h_2 = r, h_3 = r \sin \phi]$.

(c) If $x = r \cos \theta$, $y = r \sin \theta$, $z = w$ ((r, θ, w) are cylindrical coordinates) show

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial w^2}$$

[Deriving formulas like (b) directly by the chain rule is guaranteed to give you a pain in the head because of the complications in computing the second derivatives.]

5.72: (a) Let $\underset{\sim}{V}$ be the velocity field of a fluid flowing parallel to the x-axis with speed numerically equal to the distance from the yz-plane. Show that $\underset{\sim}{\nabla} \cdot \underset{\sim}{V} = 1$ and $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$.

   (b) Let $\underset{\sim}{V}$ be the velocity field of a rigid body rotating about the direction $\underset{\sim}{k}$ with angular velocity ω. Show $\underset{\sim}{\nabla} \cdot \underset{\sim}{V} = 0$ and $\underset{\sim}{\nabla} \times \underset{\sim}{V} = 2 \omega \underset{\sim}{k}$.

Exact Differential forms:

<u>Definition</u>: A k-form ω is exact in a region (open connected set) K if there exists a C^1 (k-1)-form α such that $\omega = d\alpha$.

   For example $\omega = A \, dx + B \, dy + C \, dz$ is exact if there is a $f \in C^1$ such that $\frac{\partial f}{\partial x} = A$, $\frac{\partial f}{\partial y} = B$, $\frac{\partial f}{\partial z} = C$.

<u>Theorem 5.4</u>: Let ω be an exact differential k-form and D_1, D_2 oriented k-surfaces such that $\partial D_1 = \partial D_2$ then $\omega(D_1) = \omega(D_2)$ if the GGS Theorem is applicable to D_1 and D_2. [Thus in a region in which ω is exact $\omega(D)$ depends only on ∂D].

Proof: $\quad \omega(D_1) = d\alpha(D_1) = \alpha(\partial D_1)$

$\qquad\qquad \omega(D_2) = d\alpha(D_2) = \alpha(\partial D_2)$

$\therefore \quad \omega(D_1) = \omega(D_2)$
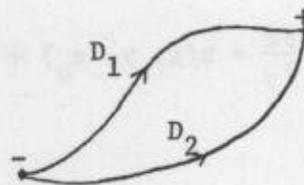
For example if $\quad \omega = A\,dx + B\,dy \quad$ and $\quad \exists\ f \in C^1 \ni \dfrac{\partial f}{\partial x} = A, \ \dfrac{\partial f}{\partial y} = B$

then $\quad \omega(\gamma_1) = f(Q) - f(P)$

and $\quad \omega(\gamma_2) = f(Q) - f(P)$
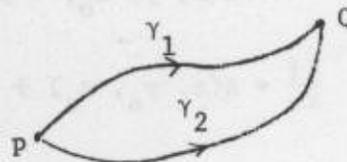
Theorem 5.5: (a) A necessary condition that $\omega(\in C^1)$ be exact in a region

$\qquad\qquad\qquad$ K is $d\omega = 0$

$\qquad\qquad$ (b) If $\omega \in C^1$ and $d\omega = 0$ in a spherical* region then ω

$\qquad\qquad\qquad$ is exact in that region. [* This may be relaxed to any

$\qquad\qquad\qquad$ simply connected region i.e. free of cavities.]

Proof of (a): ω exact $\Longleftrightarrow \omega = d\alpha \Rightarrow d\omega = d(d\alpha) = 0 \quad$ (cf. Exercise 5.55)

Proof of (b):

1-forms in R^3: Given $\omega = A\,dx + B\,dy + C\,dz$ and

$$d\omega = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right)dy\,dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right)dz\,dx + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)dx\,dy = 0$$

$$\therefore \quad \frac{\partial C}{\partial y} = \frac{\partial B}{\partial z}, \quad \frac{\partial A}{\partial z} = \frac{\partial C}{\partial x}, \quad \frac{\partial B}{\partial x} = \frac{\partial A}{\partial y}$$

Consider (cf. Exercise 4.37, p. 193)

$$f(x, y, z) = \int_{x_o}^{x} A(t, y_o, z_o)dt + \int_{y_o}^{y} B(x, t, z_o)dt + \int_{z_o}^{z} C(x, y, t)dt$$

$$\frac{\partial f}{\partial z} = C(x, y, z)$$

$$\frac{\partial f}{\partial y} = B(x, y, z_o) + \int_{z_o}^{z} \frac{\partial C}{\partial y}(x, y, t) \quad \text{(Exercise 4.30, p. 175)}$$

$$= B(x, y, z_o) + \int_{z_o}^{z} \frac{\partial B}{\partial z}(x, y, t)dt$$

$$= B(x, y, z_o) + B(x, y, z) - B(x, y, z_o) = B(x, y, z)$$

$$\frac{\partial f}{\partial x} = A(x, y_o, z_o) + \int_{y_o}^{y} \frac{\partial B}{\partial x}(x, t, z_o)dt + \int_{z_o}^{z} \frac{\partial C}{\partial x}(x, y, t)dt$$

$$= A(x, y_o, z_o) + \int_{y_o}^{y} \frac{\partial A}{\partial y}(x, t, z_o)dt + \int_{z_o}^{z} \frac{\partial A}{\partial z}(x, y, t)dt$$

$$= A(x, y_o, z_o) + A(x, y, z_o) - A(x, y_o, z_o) + A(x, y, z) - A(x, y, z_o)$$

$$= A(x, y, z) .$$

2-forms in R^3: Exercise 5.73:

[Verify that $\omega = A\,dy\,dz + B\,dz\,dx + C\,dx\,dy$, $A, B, C \in C^1$ is exact in a spherical region if

$$d\omega = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right)dx\,dy\,dz = 0 , \quad \text{cf. Buck, p. 433}]$$ □

Theorem 5.5.1: If K is a $(k+1)$ surface to which the GGS Theorem applies, $\partial K = S_1 \cup (-S_2)$ and ω is a C^1 k-form such that $d\omega = 0$ throughout K then $\omega(S_1) = \omega(S_2)$.

Proof:
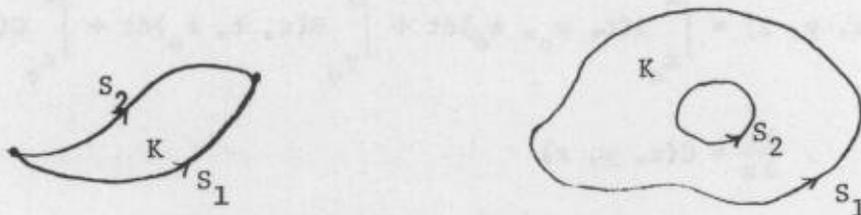
$$0 = d\omega(K) = \omega(\partial K) = \omega(S_1) - \omega(S_2)$$ □

In particular if $\omega = A\,dx + B\,dy$ and $\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y}$ throughout a region K in R^2 without cavities (simply connected) then $\omega(\gamma_1) = \omega(\gamma_2)$ for any two piecewise smooth curves γ_1, γ_2 in K with common endpoints. This is not necessarily true if the region where $d\omega = 0$ is not simply connected even if the condition fails at just a single point.

Exercises:

5.74: (a) Let $\omega = \dfrac{x}{x^2 + y^2}\,dy - \dfrac{y}{x^2 + y^2}\,dx$; $d\omega = 0$ if $(x, y) \neq (0, 0)$.

Show $\omega(\gamma) = 2\pi$ if γ is any counterclockwise oriented circle of centre $(0, 0)$.

(b) Let $\gamma[a, b]$ be a simple counterclockwise oriented closed curve in R^2 i.e. $\gamma(a) = \gamma(b)$ and $\gamma(t_1) \neq \gamma(t_2)$ if $t_1, t_2 \in (a, b)$; Show that $\omega(\gamma) = 2\pi$ if the bounded region D enclosed by γ contains $(0, 0)$ and $\omega(\gamma) = 0$ if $(0, 0)$ is in the complement of $\overline{D}$.

5.75: As a converse to Theorem 5.4 show that if D is a simply connected region in R^n and ω is a $C^1 k$-form such that the value of $\omega(K)$ depends only on the boundary of K for every k-surface K in D then ω is exact in D.

To interpret the notion of exactness for vector fields consider the force field $\underset{\sim}{F} = A\underset{\sim}{i} + B\underset{\sim}{j} + C\underset{\sim}{k}$ and the corresponding form $\omega = A\,dx + B\,dy + C\,dz$. Recall that ω is exact if there is a real valued C^1 function (0-form) f such that $\omega = df$ i.e. $\underset{\sim}{F} = \underset{\sim}{\nabla} f$ (it is a gradient field with potential f). A necessary and sufficient condition that $\underset{\sim}{F}$ be a gradient field on a simply

connected region is $d\omega = 0$ i.e. $\nabla \times F = 0$ (F is irrotational) . In particular if F is irrotational then $\omega(\gamma)$ depends only on the endpoints of γ i.e. the work done in moving from a point P to a point Q is independent of the path. Consider the velocity field $V = A\underline{i} + B\underline{j} + C\underline{k}$ and the form $\omega = A\,dy\,dz + B\,dz\,dx + C\,dx\,dy$ which is exact if $\omega = d\alpha$ for some 1-form α i.e. $V = \nabla \times F$ for some vector field F (such a V is called solenoidal). A necessary and sufficient condition that V be solenoidal on a simply connected region is $d\omega = 0$ i.e. $\nabla \cdot V = 0$ (V is source-free). For a source-free field V the flux across any surface S depends only on ∂S or equivalently the net flux across any closed surface is zero (Gauss' Theorem).

Pfaffian Differential Equations (Optional):

Given a form ω , on what surfaces S is $\omega(S) = 0$?

Example 1: Solve $\omega = 0$ if $\omega = x\,dy + y\,dx$.

$$d\omega = dx\,dy + dy\,dx = 0 \quad \therefore \quad \omega \text{ is exact}$$

$$\text{i.e. } \omega = dF \text{ for some } F$$

$$\therefore \omega(\gamma) = dF(\gamma) = F(\partial \gamma) = F(\gamma(b)) - F(\gamma(a)) = c - c = 0 \text{ on any}$$

$$\text{segment } \gamma = \gamma[a, b] \text{ of a curve } F(x,y) = c$$

$$\frac{\partial F}{\partial x} = y \qquad \frac{\partial F}{\partial y} = x$$

$$\therefore F(x,y) = xy + c_1(y) \quad \text{and} \quad F(x,y) = xy + c_2(x)$$

$$\therefore F(x,y) = xy$$

$$\therefore xy = c \text{ solves } y\,dx + x\,dy = 0$$
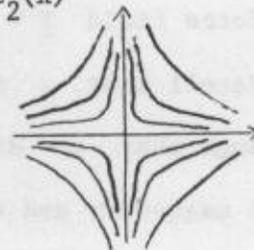
<u>Example 2:</u> $\omega = A\,dx + B\,dy + C\,dz$

If $\frac{\partial F}{\partial x} = A$, $\frac{\partial F}{\partial y} = B$, $\frac{\partial F}{\partial z} = C$ and $F \in C'$ then $\omega = 0$ on any

curve segment in the surface $F(x, y, z) = c$.

<u>Example 3:</u> $\omega = x\,dy - y\,dx$

$$d\omega = dx\,dy - dy\,dx = 2\,dx\,dy \neq 0$$

$\therefore \omega$ is not exact

But $\omega = x\,dy - y\,dx = 0$

$\Leftrightarrow \frac{1}{xy}\,\omega = \frac{dy}{y} - \frac{dx}{x} = 0$, $xy \neq 0$

This is exact and has solution set

$$F(x,y) = \log\left(\frac{y}{x}\right) = C$$

i.e. $y = kx$

$\frac{1}{xy}$ is called an integrating factor
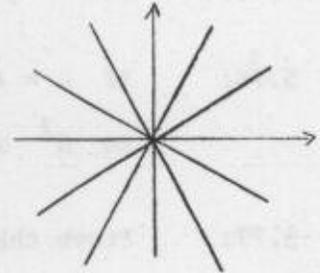
In general a o-form (real function) λ is an <u>integrating factor</u> for a k-form

ω $(k > 0)$ if $\lambda\omega$ is exact $(\lambda \neq 0)$.

$$\lambda\omega \quad \text{exact} \iff d(\lambda\omega) = 0$$

$$\iff (d\lambda)\omega + \lambda(d\omega) = 0$$

$$\iff (d\lambda)\omega\omega + \lambda(d\omega)\omega = 0$$

But $\omega\omega = 0$ (verify this) and thus, if $\lambda \neq 0$, $(d\omega)\omega = 0$. The condition

$(d\omega)\omega = 0$ is a necessary condition for the existence of an integrating

factor λ for ω. It is also a sufficient condition under very general

circumstances (not considered here).

Exercises:

5.76:   Verify that if P, Q, R are C^1 then a necessary condition for

   $\omega = P\ dx + Q\ dy + R\ dz$ to have a nontrivial integrating factor is

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0$$

   [Either check that this is the condition $(d\omega)\omega = 0$ or you may have

   done it already as Exercise 4.36, p. 193.]

5.77:   If $\omega = 2xy^3 dx + 3x^2y^2 dy$ show that $\omega(\gamma_1) = \omega(\gamma_2)$ for any two

   piecewise smooth curves in the plane with common endpoints.

5.78:   If $\omega = (3x^2y + 2xy)dx + (x^3 + x^2 + 2y)dy$ show that ω is exact

   in R^2 and find f such that $df = \omega$.

5.79:   Prove that the following integrals are independent of the path of

   integration and find their values.

   (i) $\displaystyle\int_{(0,0)}^{(a,b)} x\ dy + y\ dx$

   (ii) $\displaystyle\int_{(0,0,0)}^{(a,b,c)} (x^2 - yz)dx + (y^2 - zx)dy + (z^2 - xy)dz$

References for Chapter V

R. C. Buck: Advanced Calculus, Chapters 6, 7.

R. Courant: Differential and integral Calculus, Vol. II, Chapters IV, V.

H. Sagan: Advanced Calculus, Chapters 11, 12, 13.

CHAPTER SIX

INFINITE SERIES AND IMPROPER INTEGRALS

INFINITE SERIES

Let $a_k \in R$, $k = 1, 2, \ldots$.

Definition:

(i) $\sum\limits_{k=1}^{\infty} a_k$ is said to be <u>convergent</u> ($\sum\limits_{k=1}^{\infty} a_k \in C$) and have <u>sum</u>

S if

$$\lim_{n \to \infty} S_n \ \exists\ = S$$

where $S_n = \sum\limits_{k=1}^{n} a_k$. S_n is called a <u>partial sum</u> of the series.

(ii) $\sum\limits_{k=1}^{\infty} a_k$ is said to be <u>divergent</u> ($\sum\limits_{k=1}^{\infty} a_k \in D$) if

$\lim\limits_{n \to \infty} S_n \not\exists$ (i.e. $\{S_n\}$ divergent)

Example 1: What does it mean to say $\frac{1}{3} = 0.333\ldots$?

$$\sum_{k=1}^{\infty} \frac{3}{10^k} = \frac{1}{3}$$

since $S_n = \sum\limits_{k=1}^{n} \frac{3}{10^k} = \frac{3}{10} (1 + \frac{1}{10} + \cdots + \frac{1}{10^{k-1}})$

$= \frac{3}{10} (1 - \frac{1}{10^n}) / (1 - \frac{1}{10})$

$= \frac{1}{3} (1 - \frac{1}{10^n})$

$\therefore \lim\limits_{n \to \infty} S_n = \frac{1}{3}$

Notice that $\sum\limits_{k=1}^{\infty} a_k = S \Leftrightarrow \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_k = S \Leftrightarrow$ for each $\epsilon > 0$,

$\exists$ N $\ni$ if $n \geq N$ then

$$\left| \sum_{k=1}^{n} a_k - S \right| < \epsilon$$

Therefore, from Theorem 2.7 page 53, we have the following theorem.

Theorem 6.1 (Cauchy Criterion):

$$\sum_{k=1}^{\infty} a_k \in C \Leftrightarrow \text{for each } \epsilon > 0, \exists \text{ N} \ni \text{if } m \geq n \geq N \text{ then}$$

$$\left| \sum_{k=n}^{m} a_k \right| < \epsilon \quad (\text{i.e. } |S_m - S_{n-1}| < \epsilon) . \qquad \Box$$

Example 2: $\sum\limits_{k=1}^{\infty} (-1)^k \in D$

For all n $|S_{n+1} - S_n| = |a_{n+1}| = 1$

Example 3: $\sum\limits_{k=1}^{\infty} \frac{1}{k} \in D$

For all n $\left| \sum\limits_{k=n}^{2n} \frac{1}{k} \right| = \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n} \geq \frac{n}{2n} = \frac{1}{2}$

Exercise:

6.1: If $\sum\limits_{k=1}^{\infty} a_k = S$, $\sum\limits_{k=1}^{\infty} b_k = T$ and $\alpha, \beta \in R$ then

$$\sum_{k=1}^{\infty} (\alpha a_k + \beta b_k) = \alpha S + \beta T .$$

__Theorem 6.2:__ $\sum\limits_{k=1}^{\infty} a_k \in C \Rightarrow \lim\limits_{n \to \infty} a_n = 0$

Proof:

$$\lim\limits_{n \to \infty} S_n = S \qquad\qquad \lim\limits_{n \to \infty} S_{n-1} = S$$

$$\therefore \quad \lim\limits_{n \to \infty} (S_n - S_{n-1}) = S - S = 0$$

i.e. $\lim\limits_{n \to \infty} a_n = 0$, since $S_n - S_{n-1} = a_n$

__Example 4:__

$$\sum\limits_{k=0}^{\infty} x^k \begin{cases} = \dfrac{1}{1-x} , & \text{if } |x| < 1 \\[2mm] \in D , & \text{if } |x| \geq 1 \end{cases}$$

$$S_n = 1 + x + \dots + x^n$$

$$xS_n = x + x^2 + \dots + x^{n+1}$$

$$(1 - x)S_n = 1 - x^{n+1}$$

$$\therefore \quad S_n = \frac{1 - x^{n+1}}{1 - x} , \quad x \neq 1$$

$$\lim\limits_{n \to \infty} S_n \begin{cases} = \dfrac{1}{1-x} , & |x| < 1 \\[2mm] \neq| & , |x| \geq 1 \end{cases}$$

__Example 5:__

$$\sum\limits_{k=1}^{\infty} (-1)^k \in D \quad \text{since} \quad S_n = \begin{cases} -1 , & n \text{ odd} \\[2mm] 0 , & n \text{ even} \end{cases}$$

$$\therefore \lim\limits_{n \to \infty} S_n \neq|$$

Example 6:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$$

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\therefore \quad S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = (\frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{n(n+1)})$$

$$= (1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + \ldots + \frac{1}{n} - \frac{1}{n+1})$$

$$= (1 - \frac{1}{n+1})$$

$$\therefore \quad \lim_{n \to \infty} S_n = 1$$

Remarks: (i) $\lim\limits_{n \to \infty} a_n = 0 \not\Rightarrow \sum\limits_{k=1}^{\infty} a_k \in C$

e.g. $\sum\limits_{k=1}^{\infty} \frac{1}{k} \in D$

(ii) $\lim\limits_{n \to \infty} a_n \neq 0 \Rightarrow \sum\limits_{k=1}^{\infty} a_k \in D$

(i.e. if either $\lim\limits_{n \to \infty} a_n \neq |$ or $\exists | \neq 0$ then $\sum\limits_{k=1}^{\infty} a_k \in D$)

Series of positive terms:

Theorem 6.3: If $a_k \geq 0$, $k = 1, 2, \ldots$, then $\sum\limits_{k=1}^{\infty} a_k \in C \iff \{ \sum\limits_{k=1}^{n} a_k \}$ is a bounded sequence.

Proof: Theorem 2.6 page 50. $S_{n+1} = S_n + a_{n+1} \geq S_n$. $\therefore \{S_n\}$ increasing so it is convergent if and only if it is bounded. □

Corollary 6.3.1 (Comparison Test, important):

If $0 \leq a_k \leq b_k$, $k = 1, 2, \ldots$, then

(i) $\sum_{k=1}^{\infty} a_k \in D \Rightarrow \sum_{k=1}^{\infty} b_k \in D$

(ii) $\sum_{k=1}^{\infty} b_k \in C \Rightarrow \sum_{k=1}^{\infty} a_k \in C$

Proof of (i): Let $S_n = \sum_{k=1}^{n} a_k$, $T_n = \sum_{k=1}^{n} b_k$, $0 \leq S_n \leq T_n$.

$$\sum_{k=1}^{\infty} a_k \in D \Rightarrow \{S_n\} \text{ unbounded}$$

$$\Rightarrow \{T_n\} \text{ unbounded}$$

$$\Rightarrow \sum_{k=1}^{\infty} b_k \in D. \qquad \square$$

Proof of (ii): Exercise 6.2 .

Corollary 6.3.2 (Comparison Test):

If $a_k \geq 0$, $b_k > 0$ $k = 1, 2, \ldots$ and $\lim_{k \to \infty} \frac{a_k}{b_k} = L$, $0 < L < \infty$,

then

$$\sum_{k=1}^{\infty} a_k \in C \iff \sum_{k=1}^{\infty} b_k \in C$$

Proof: For all sufficiently large k $0 < \frac{L}{2} b_k < a_k < 2L b_k$. $\qquad \square$

Exercise:

6.3: Show that "$\Leftarrow$" holds but not necessarily "$\Rightarrow$" if $L = 0$.

Example: We have seen $\sum\limits_{k=1}^{\infty} \dfrac{1}{k(k+1)} \in C$ $\therefore$ $\sum\limits_{k=1}^{\infty} \dfrac{1}{k^2} \in C$ since $\lim\limits_{k\to\infty} \dfrac{k^2}{k(k+1)} = 1$

Corollary 6.3.3 (Ratio Comparison Test):

If $a_n > 0$, $b_n > 0$ and $\dfrac{a_{n+1}}{a_n} \le \dfrac{b_{n+1}}{b_n}$, $\forall\, n \ge N$, then

(i) $\sum\limits_{k=1}^{\infty} a_k \in D \Rightarrow \sum\limits_{k=1}^{\infty} b_k \in D$

(ii) $\sum\limits_{k=1}^{\infty} b_k \in C \Rightarrow \sum\limits_{k=1}^{\infty} a_k \in C$

Proof:

$$\dfrac{a_{n+1}}{a_n} \le \dfrac{b_{n+1}}{b_n} \quad, \quad a_n, b_n > 0, \quad n \ge N$$

$$\Rightarrow \dfrac{a_{n+1}}{b_{n+1}} \le \dfrac{a_n}{b_n}, \quad n \ge N$$

i.e. $\left\{ \dfrac{a_n}{b_n} \right\}$ decreasing $\quad n \ge N$

$$\therefore \quad \dfrac{a_n}{b_n} \le \dfrac{a_N}{b_N} = M, \quad \text{if } n \ge N$$

$$\therefore \quad 0 < a_n \le M\, b_n, \quad \text{if } n \ge N. \qquad \square$$

Corollary 6.3.4 (Ratio Test): $a_n > 0$, $n = 1, 2, \ldots$.

(i) If $\dfrac{a_{n+1}}{a_n} \le x < 1$, $\forall\, n \ge N$, then $\sum\limits_{k=1}^{\infty} a_k \in C$

(ii) If $\dfrac{a_{n+1}}{a_n} \ge x \ge 1$, $\forall\, n \ge N$, then $\sum\limits_{k=1}^{\infty} a_k \in D$

Proof: Choose $b_n = x^n$, $\dfrac{b_{n+1}}{b_n} = x$

$$\text{and} \quad \sum_{k=1}^{\infty} b_k \in \begin{cases} C \text{, if } |x| < 1 \text{,} \\ \\ D \text{, if } |x| \geq 1 \text{.} \end{cases}$$

<u>Corollary 6.3.5</u> (Ratio Test): $a_n > 0$, $n = 1, 2, \ldots$.

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = c$$

(i) $0 \leq c < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \in C$

(ii) $c > 1 \quad \Rightarrow \sum_{n=1}^{\infty} a_n \in D$

(iii) $c = 1 \quad \Rightarrow \quad ? \quad$ (Test fails)

<u>Examples</u>:

$$0 \leq c < 1 : \sum_{k=1}^{\infty} \frac{1}{k!} \in C \text{ since } \frac{a_{n+1}}{a_n} = \frac{1}{n+1} \to 0$$

$$c > 1 \quad : \sum_{k=1}^{\infty} k! \in D \text{ since } \frac{a_{n+1}}{a_n} = n+1 \to \infty$$

$$c = 1 \quad : \sum_{k=1}^{\infty} \frac{1}{k^2} \in C \text{ and } \frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} \to 1$$

$$\text{but} \quad \sum_{k=1}^{\infty} \frac{1}{k} \in D \text{ and } \frac{a_{n+1}}{a_n} = \frac{n}{n+1} \to 1$$

Corollary 6.3.6 (Root Test): $a_n \geq 0$, $n = 1, 2, \ldots$.

    (i) If $a_n^{1/n} \leq x < 1$, $\forall\, n \geq N$ then $\sum\limits_{k=1}^{\infty} a_k \in C$

    (ii) If $a_n^{1/n} \geq x \geq 1$, $\forall\, n \geq N$ then $\sum\limits_{k=1}^{\infty} a_k \in D$

    In particular, if $\lim\limits_{n \to \infty} a_n^{1/n} = c$, then

        (i) $0 \leq c < 1$ $\Rightarrow$ $\sum\limits_{k=1}^{\infty} a_k \in C$

        (ii) $c > 1$ $\Rightarrow$ $\sum\limits_{k=1}^{\infty} a_k \in D$

        (iii) $c = 1$ $\Rightarrow$?

Proof: Exercise 6.4 .

Exercises:

6.5:   Show (i) $\sum\limits_{k=1}^{\infty} \dfrac{5}{7^{k-1}} = \dfrac{35}{6}$, (ii) $\sum\limits_{k=1}^{\infty} \dfrac{\pi}{\sqrt{2}^{k-1}} = \dfrac{\pi\sqrt{2}}{\sqrt{2} - 1}$

        (iii) $\sum\limits_{k=1}^{\infty} \left(\dfrac{-1}{4} \right)^{k-1} = \dfrac{4}{5}$ (iv) $0.\overline{612}612612\ldots = \dfrac{68}{111}$

6.6:   Show that each of the following is divergent:

        (i) $\sum\limits_{k=1}^{\infty} \dfrac{k}{k+1}$ (ii) $\sum\limits_{k=1}^{\infty} \dfrac{1}{\sqrt{k}}$

        (iii) $\sum\limits_{k=1}^{\infty} \left(\dfrac{3}{2} \right)^{k}$ (iv) $\sum\limits_{k=2}^{\infty} \dfrac{1}{\log k}$

6.7:   Show

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{(k+1)!} = 1$ (ii) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}$

(iii) $\displaystyle\sum_{k=1}^{\infty} \frac{(k-1)!}{(k+p)!} = \frac{1}{p(p!)}$

6.8:   Show $\displaystyle\sum_{k=1}^{\infty} k\, x^k = \frac{x}{(1-x)^2}$, if $|x| < 1$.

<u>Theorem 6.4</u> (Integral Test): Suppose f is continuous <u>nonincreasing</u> and <u>positive</u> on $[1,\infty)$ then

$$\sum_{k=1}^{\infty} f(k) \in C \iff \int_1^{\infty} f \; \exists\,|$$

$$\left[\text{where } \int_1^{\infty} f \stackrel{\text{def}}{=} \lim_{T\to\infty} \int_1^T f \,\right]$$

Proof: $f(k) \geq f(x) \geq f(k+1)$

$\therefore$ $f(k)\cdot 1 \geq \displaystyle\int_k^{k+1} f \geq f(k+1)\cdot 1$

$\therefore$ $\displaystyle\sum_{k=1}^{n} f(k) \geq \int_1^{n+1} f \geq \sum_{k=1}^{n} f(k+1) = \sum_{k=2}^{n+1} f(k)$

i.e. $S_n \geq \displaystyle\int_1^{n+1} f \geq S_{n+1} - f(1)$. $\square$

Example:

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \in \begin{cases} D \text{ , if } p \leq 1 \\ \\ C \text{ , if } p > 1 \end{cases}$$

$p \leq 0 : \quad D$ since $\frac{1}{n^p} \not\to 0 \quad (n \to \infty)$

$p \geq 0 : \quad \frac{1}{k^p}$ is decreasing

$$\int_1^T \frac{1}{x^p} \, dx = \begin{cases} \frac{1}{1-p} (T^{1-p} - 1) \text{ , } p \neq 1 \\ \\ \log T \qquad\qquad \text{ , } p = 1 \end{cases}$$

$$\therefore \quad \int_1^\infty \frac{1}{x^p} \, dx = \begin{cases} \frac{1}{p-1} \text{ , } p > 1 \\ \\ \not\exists \text{ , } p \leq 1 \end{cases}$$

In the limit form of the Ratio Test (Corollary 6.3.5) the following sometimes works in the critical case $\lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n} = 1$

Corollary 6.4.1 (Raabe's Test): $a_n \geq 0$, $n = 1, 2, \ldots$.

(i) If $\dfrac{a_{n+1}}{a_n} \leq 1 - \dfrac{p}{n}$, $\forall n \geq N$ and $p > 1$ then $\sum\limits_{k=1}^{\infty} a_k \in C$.

(ii) If $\dfrac{a_{n+1}}{a_n} \geq 1 - \dfrac{p}{n}$, $\forall n \geq N$ and $p \leq 1$ then $\sum\limits_{k=1}^{\infty} a_k \in D$.

We shall need the observation that if $p > 1$ and $n > 0$ then

$(1 - \frac{p}{n}) < (1 - \frac{1}{n})^p$. This follows from the fact that if $0 < x < 1$

$$1 - x^P = (1-x)pc^{p-1}, \quad x < c < 1 \quad \text{(MVTh)}$$

$$< (1-x)p$$

$$\therefore \quad 1 - \left(1 - \frac{1}{n}\right)^P < \frac{p}{n} \quad \text{i.e.} \quad 1 - \frac{p}{n} < \left(1 - \frac{1}{h}\right)^P .$$

Proof of (i): Choose $b_n = \dfrac{1}{(n-1)^P}, p > 1 ; \quad \sum_{k=2}^{\infty} b_n \in C$

and $\dfrac{b_{n+1}}{b_n} = \dfrac{(n-1)^P}{n^P} = \left(1 - \dfrac{1}{n}\right)^P > 1 - \dfrac{p}{n} \geq \dfrac{a_{n+1}}{a_n}$

$$\Rightarrow \sum_{k=1}^{\infty} a_k \in C$$

Proof of (ii): Choose $b_n = \dfrac{1}{n-1} : \quad \sum_{k=2}^{\infty} b_n \in D$

and $\dfrac{b_{n+1}}{b_n} = \dfrac{n-1}{n} = 1 - \dfrac{1}{n} \leq 1 - \dfrac{p}{n} \quad (p \leq 1)$

$$\leq \frac{a_{n+1}}{a_n} \quad \Rightarrow \sum_{k=1}^{\infty} a_k \in D \qquad \qquad \square$$

Exercise:

6.9: Show that if

$$\frac{a_{n+1}}{a_n} \geq 1 - \frac{1}{n} - \frac{A}{n^2}$$

for any A and all $n \geq N$ then $\displaystyle\sum_{k=1}^{\infty} a_k \in D$

[Consider $\dfrac{b_{n+1}}{b_n}$ where $b_n = \dfrac{1}{n-A-1} ; \quad \sum b_k \in D$] .

Example 1:

We have already seen that $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \in C$ by showing that its

sum is 1. However the Ratio Test fails here:

$$\frac{a_{n+1}}{a_n} = \frac{n(n+1)}{(n+1)(n+2)} = \frac{n}{n+2} \to 1 \quad \text{(Critical Case)}$$

But $\quad \frac{a_{n+1}}{a_n} = \frac{n}{n+2} = 1 - \frac{2}{n+2} \leq 1 - \frac{3}{2n}, \quad n > 6,$

$$\Rightarrow \sum_{k=1}^{\infty} a_k \in C \quad \text{by Raabe's Test .}$$

Alternatively: $\quad \frac{a_{n+1}}{a_n} = \frac{n}{n+2} = \frac{1}{1 + \frac{2}{n}} = 1 - \frac{2}{n} + R_2(n)$

$$\leq 1 - \frac{3}{2n} , \text{ if n is sufficiently large,}$$

since $|R_2(n)| \leq \frac{C}{n^2}$ (Taylor's Theorem) $\frac{3}{2} > 1 \Rightarrow \sum_{k=1}^{\infty} a_k \in C$ by Raabe's Test

Example 2:

$$\sum_{k=1}^{\infty} \frac{a(a+1) \, \ldots \, (a+k) \, b(b+1) \, \ldots \, (b+k)}{(k+1)! \, c(c+1) \, \ldots \, (c+k)} \quad (c \neq -n)$$

$$\in \begin{cases} C, & \text{if } c > a + b \\ \\ D, & \text{if } c \leq a + b \end{cases}$$

$$\frac{a_{n+1}}{a_n} = \frac{(a+n+1)(b+n+1)}{(n+2)(c+n+1)} = \frac{(1 + \frac{a+1}{n})(1 + \frac{b+1}{n})}{(1 + \frac{2}{n})(1 + \frac{c+1}{n})} \to 1$$

so the Ratio Test fails .

But $\dfrac{a_{n+1}}{a_n} = (1 + \dfrac{a+1}{n})(1 + \dfrac{b+1}{n})(1 + \dfrac{2}{n})^{-1}(1 + \dfrac{c+1}{n})^{-1}$

$= (1 + \dfrac{a+b+2}{n} + 0\,(\dfrac{1}{n^2}))(1 - \dfrac{2}{n} + 0\,(\dfrac{1}{n^2}))(1 - \dfrac{c+1}{n} + 0\,(\dfrac{1}{n^2}))$

$= 1 - \dfrac{1+c-a-b}{n} + 0\,(\dfrac{1}{n^2})$

where $0\,(\dfrac{1}{n^2})$ denotes an expression $R(n)$ such that $|R(n)| \le \dfrac{C}{n^2}$ for all

large n .

$$\therefore \sum_{k=1}^{\infty} a_k \in \begin{cases} C & \text{if } 1 + c - a - b > 1 \\ \\ D & \text{if } 1 + c - a - b \le 1 \end{cases}$$

Note: In the case $1 + c - a - b = 1$ you need Exercise 6.9.

Example 3:

$$\sum_{k=1}^{\infty} \left[\dfrac{1.3.5 \ldots (2k-1)}{2.4.6 \ldots 2k} \right]^P \in \begin{cases} C, & \text{if } p > 2 \\ \\ D, & \text{if } p \le 2 \end{cases}$$

$\dfrac{a_{n+1}}{a_n} = (\dfrac{2n+1}{2n+2})^P = (1 + \dfrac{1}{2n})^P (1 + \dfrac{1}{n})^{-P} \to 1$ (Ratio Test fails)

$\dfrac{a_{n+1}}{a_n} = (1 + \dfrac{P}{2n} + 0\,(\dfrac{1}{n^2}))(1 - \dfrac{P}{n} + 0\,(\dfrac{1}{n^2}))$

$= 1 - \dfrac{P}{2n} + 0\,(\dfrac{1}{n^2})$

$$\therefore \sum_{k=1}^{\infty} a_k \in \begin{cases} C, & \text{if } \dfrac{P}{2} > 1 \\ \\ D, & \text{if } \dfrac{P}{2} \le 1 \end{cases}$$

Example 4:

$$\sum_{k=1}^{\infty} \frac{2^k}{k(k+1)} \in D$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{2^n} = \frac{2n}{n+2} \to 2 > 1$$

$$\therefore \sum_{k=1}^{\infty} a_k \in D \quad \text{(Ratio Test)} .$$

Example 5:

$$\sum_{k=1}^{\infty} \frac{1}{2^k k(k+1)} \in C \qquad \text{since}$$

$$0 \le \frac{1}{2^n n(n+1)} \le \frac{1}{2^n} \quad \text{and} \quad \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \in C$$

$$\underline{\underline{\text{or}}} \quad 0 \le \frac{1}{2^n n(n+1)} \le \frac{1}{n(n+1)} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \in C$$

$$\underline{\underline{\text{or}}} \quad \frac{a_{n+1}}{a_n} = \frac{n}{2(n+2)} \to \frac{1}{2} < 1 \quad \text{(Ratio Test)} .$$

Exercises:

6.10: Show $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\log k)^p} \in \begin{cases} D , p \le 1 \\ \\ C , p > 1 \end{cases}$

6.11: Show

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{3k+2}{3.6 \cdots (3k+3)} \in C$,

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{5^k 2^{k+1}} \in C$,

(iii) $\displaystyle\sum_{k=0}^{\infty} \left[\frac{1.3 \cdots (2k+1)}{2.4 \cdots (2k+2)}\right]^3 \in C$,

(iv) $\displaystyle\sum_{k=0}^{\infty} \left[\frac{2.4 \cdots (2k+2)}{1.3 \cdots (2k+1)}\right]^3 \in D$,

(v) $\sum_{k=0}^{\infty} \frac{1+2k^2}{1+k^2} \in D$, (vi) $\sum_{k=0}^{\infty} \frac{k^{100}}{k!} \in C$.

6.12 (Euler's constant): Show $\gamma = \lim_{n \to \infty} [1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n]$ exists

[If $c_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n$, $c_n \geq c_{n+1} > 0$]

6.13: Show

(i) $\sum_{k=0}^{\infty} \frac{1}{(\alpha+k)(\alpha+k+1)} = \frac{1}{\alpha}$ $(\alpha \neq 0)$, (ii) $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} = \frac{1}{4}$

6.14:   A sequence of real numbers $\{a_n\}$ is said to be of <u>bounded variation</u>

(BV) if $\sum_{k=1}^{\infty} |a_{k+1} - a_k| \in C$.

(i) Show that $\{a_n\} \in BV \Rightarrow \{a_n\}$ is bounded.

(ii) Show that $\{a_n\} \in BV \Rightarrow \{a_n\}$ is convergent.

(iii) A bounded increasing sequence is of bounded variation.

(iv) The sum of a bounded increasing sequence and a bounded decreasing sequence is of bounded variation.

(v) A sequence of bounded variation is the sum of a bounded increasing sequence and a bounded decreasing sequence

[Consider $\pi_n = \frac{1}{2}(a_n + \sum_{k=1}^{n-1} |a_{k+1} - a_k|)$, $\nu_n = \frac{1}{2}(a_n - \sum_{k=1}^{n-1} |a_{k+1} - a_k|)$].

6.15:   Let $a_n \geq 0$.

(i) If $p > 0$ and $\frac{a_{n+1}}{a_n} \leq 1 - \frac{p}{n}$ for all sufficiently large n   then

$$\lim_{n \to \infty} a_n = 0 .$$

(ii) If $p \leq 0$ and $\frac{a_{n+1}}{a_n} \geq 1 - \frac{p}{n}$ for all sufficiently large n

then $\{a_n\}$ is divergent (in fact $\lim_{n\to\infty} a_n = \infty$).

(iii) Parts (i), (ii) may sometimes resolve the critical case (c) of the

Ratio Test for sequences (Exercise 2.10 p. 48).

If $a_n = \frac{1.3 \ldots (2n+1)}{2.4 \ldots (2n+2)}$ show $\lim_{n\to\infty} a_n = 0$. Is $\sum_{k=1}^{\infty} a_k$ convergent?

6.16: If $a_n \geq 0$ then $\sum_{k=1}^{\infty} a_k \in C \Rightarrow \sum_{k=1}^{\infty} \sqrt{a_k} \; k^{-p} \in C$ if $p > \frac{1}{2}$. Give a

counterexample for the case $p = \frac{1}{2}$.

Absolute convergence and conditional convergence:

We shall no longer assume $a_n \geq 0$.

__Definition:__ $\sum_{k=1}^{\infty} a_k$ is __absolutely__ __convergent__ ($\sum_{k=1}^{\infty} a_k \in Abs \; C$)

if $\sum_{k=1}^{\infty} |a_k|$ is convergent .

__Theorem 6.5:__ $\sum_{k=1}^{\infty} a_k \in Abs \; C \Rightarrow \sum_{k=1}^{\infty} a_k \in C$

Proof: $\sum_{k=1}^{\infty} a_k \in Abs \; C \Rightarrow \sum_{k=1}^{\infty} |a_k| \in C$

$\Rightarrow \forall \; \varepsilon > 0 , \; \exists \; N \ni \text{if } m,n \geq N \text{ then } \sum_{k=m}^{n} |a_k| < \varepsilon$

$\Rightarrow \text{if } m,n \geq N \text{ then } |\sum_{k=n}^{m} a_k| \leq \sum_{k=m}^{n} |a_k| < \varepsilon$

$\therefore \; \sum_{k=1}^{\infty} a_k \in C$ (Cauchy Criterion).

Examples:

1. $\displaystyle\sum_{k=0}^{\infty} x^k \in Abs \ C$ if $|x| < 1$

$$\text{since} \quad \sum_{k=0}^{\infty} |x^k| = \sum_{k=0}^{\infty} |x|^k = \frac{1}{1-|x|} \quad \text{if } |x| < 1 \ .$$

2. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k^2} \in Abs \ C$

3. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k} \notin Abs \ C$. However we will see that it is convergent.

4. $\displaystyle\sum_{k=0}^{\infty} \frac{\sin k}{k^2} \in Abs \ C$.

Tests for nonabsolute convergence:

Lemma 6.6.1 (Abel's Lemma):

$$\sum_{k=n}^{m} u_k (v_{k+1} - v_k) = u_{m+1} v_{m+1} - u_n v_n - \sum_{k=n}^{m} v_{k+1}(u_{k+1} - u_k)$$

This is a summation by parts formula.

Proof:

$$\sum_{k=n}^{m} u_k(v_{k+1} - v_k) = \sum_{k=n}^{m} u_k v_{k+1} - \sum_{k=n}^{m} u_k v_k$$

$$= u_{m+1} v_{m+1} - u_n v_n + \sum_{k=n}^{m} u_k v_{k+1} - \sum_{k=n+1}^{m+1} u_k v_k$$

$$= u_{m+1} v_{m+1} - u_n v_n + \sum_{k=n}^{m} u_k v_{k+1} - \sum_{k=n}^{m} u_{k+1} v_{k+1}$$

$$= u_{m+1} v_{m+1} - u_n v_n - \sum_{k=n}^{m} v_{k+1}(u_{k+1} - u_k)$$

<u>Theorem 6.6</u> (Dirichlet's Test): Suppose

(i) $\{ \sum_{k=1}^{n} b_k \}$ is a bounded sequence,

(ii) $\lim_{n \to \infty} a_n = 0$ and $\sum_{k=1}^{\infty} |a_{k+1} - a_k| \in C.$

then $\sum_{k=1}^{\infty} a_k b_k \in C .$

[(ii) holds in particular if $\{a_n\}$ is monotone and $\lim_{n \to \infty} a_n = 0$]

<u>Remark:</u> A sequence $\{a_n\}$ for which $\sum_{k=1}^{\infty} |a_{k+1} - a_k| \in C$ is said to be of

<u>bounded variation</u> (BV). In particular a monotone convergent sequence is of bounded variation. A sequence is of bounded variation if and only if it is the difference of two monotone increasing sequences (cf. Exercise 6.14).

Proof of Theorem 6.6: Let $S_n = \sum_{k=1}^{n} b_k$; (i) $=> |S_n| \le B, \forall n .$

$$\therefore \sum_{k=n}^{m} a_k b_k = \sum_{k=n}^{m} a_k (S_k - S_{k-1})$$

$$= a_{m+1} S_m - a_n S_{n-1} - \sum_{k=n}^{m} S_k (a_{k+1} - a_k)$$

$$\therefore |\sum_{k=n}^{m} a_k b_k| \le |a_{m+1}|B + |a_n|B + B \sum_{k=n}^{m} |a_{k+1} - a_k| \qquad (*)$$

Since $\lim_{n \to \infty} a_n = 0$ and $\sum_{k=1}^{\infty} |a_{k+1} - a_k| \in C$, if $\epsilon > 0,$ then

$|a_{m+1}| < \frac{\epsilon}{3B} \quad |a_n| < \frac{\epsilon}{3B}$ and $\sum_{k=n}^{m} |a_{k+1} - a_k| < \frac{\epsilon}{3B}$ if m and n are large

enough i.e. $m, n \geq N$ for some N.

$$\therefore \; |\sum_{k=n}^{m} a_k b_k| < 3(\frac{\epsilon}{3B})B = \epsilon \quad \text{if } m,n \geq N \quad \text{(by }^*\text{)}.$$

$$\therefore \; \text{Cauchy criterion} \Rightarrow \sum_{k=1}^{\infty} a_k b_k \in C. \qquad \square$$

<u>Example</u>: $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \in C$

Let $a_k = \frac{1}{k}, \quad b_k = (-1)^k$

(i) $\sum_{k=1}^{n} b_k = \begin{cases} -1, & n \text{ odd} \\ & \quad\quad\quad \text{is bounded} \\ 0, & n \text{ even} \end{cases}$

(ii) $a_n \downarrow 0$ (a_n decreases to 0).

$$\therefore \; \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \in C.$$

<u>Corollary 6.6.1</u> (Abel's Test): Suppose

(i) $\sum_{k=1}^{\infty} b_k \in C$,

(ii) $\sum_{k=1}^{\infty} |a_{k+1} - a_k| \in C$.

Then $\sum_{k=1}^{\infty} a_k b_k \in C$.

[(ii) holds in particular if $\{a_n\}$ is monotone and bounded.]

Proof of Corollary 6.6.1: A direct proof may be given based on Abel's Lemma. The following shows it is implied by Dirichlet's Test.   Let $A_k = a_k - a$ where $a = \lim\limits_{n \to \infty} a_n$ which exists by Exercise 6.14.

(i) $\sum\limits_{k=1}^{\infty} b_k \in C \implies \{\sum\limits_{k=1}^{n} b_k\}$ is bounded.

(ii) $\sum\limits_{k=1}^{\infty} |A_{k+1} - A_k| = \sum\limits_{k=1}^{\infty} |a_{k+1} - a_k| \in C$ and $\lim\limits_{n \to \infty} A_n = 0$

$\therefore \sum\limits_{k=1}^{\infty} A_k b_k \in C$ by Dirichlet's Test .

i.e. $\sum\limits_{k=1}^{\infty} (a_k - a) b_k \in C$

$\therefore \sum\limits_{k=1}^{\infty} b_k \in C \implies \sum\limits_{k=1}^{\infty} a b_k \in C$

$\implies \sum\limits_{k=1}^{\infty} [(a_k - a) b_k + a b_k] \in C$

i.e. $\sum\limits_{k=1}^{\infty} a_k b_k \in C$ □

Corollary 6.6.2 (Leibniz alternating series test):

$$a_k \downarrow 0 \implies \sum\limits_{k=1}^{\infty} (-1)^k a_k \in C$$

Proof:

(i) $b_k = (-1)^k, \quad \sum\limits_{k=1}^{n} (-1)^k = \begin{cases} -1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

$\therefore \{\sum\limits_{k=1}^{n} b_k\}$ bounded

(ii) $a_k \downarrow 0$

so the result follows from Dirichlet's Test. □

<u>Examples</u>:

(1) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\log k} \in C$ since $\dfrac{1}{\log k} \downarrow 0$

$\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\log k} \notin$ Abs C (Why?)

(2) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p} \in \begin{cases} C \iff p > 0 \\ \\ \text{Abs } C \iff p > 1 \end{cases}$

(3) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{\log k}{k} \in C$, $\notin$ Abs C .

(4) $1 - \dfrac{1}{2^2} + \dfrac{1}{2} - \dfrac{1}{3^2} + \dfrac{1}{3} - \ldots - \dfrac{1}{n^2} + \dfrac{1}{n} - \ldots \in D$

$S_{2n} = (1 + \dfrac{1}{2} + \ldots + \dfrac{1}{n}) - (\dfrac{1}{2^2} + \ldots + \dfrac{1}{n^2})$

and $\displaystyle\lim_{n\to\infty} (1 + \dfrac{1}{2} + \ldots + \dfrac{1}{n}) = \infty$, $\displaystyle\lim_{n\to\infty} (\dfrac{1}{2^2} + \ldots + \dfrac{1}{n^2}) < \infty$.

What condition of Leibniz's Test fails to hold here?

(5) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos kx}{k} \in \begin{cases} D , & x = 2 m \pi \\ \\ C , & x \neq 2 m \pi \end{cases}$

(i) $b_k = \cos kx$

$\displaystyle\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} \cos kx = \frac{\sin(n + \frac{1}{2})x - \sin \frac{1}{2} x}{2 \sin \frac{1}{2} x}$

since $2 \sin \frac{1}{2} x \cos kx = \sin (k + \frac{1}{2})x - \sin (k - \frac{1}{2})x$

$\therefore \quad 2 \sin \frac{1}{2} x \sum_{k=1}^{n} \cos kx = \sum_{k=1}^{n} \sin(k + \frac{1}{2})x - \sin(k - \frac{1}{2})x$

$$= \sin(n + \frac{1}{2})x - \sin \frac{1}{2} x$$

Thus $\sum_{k=1}^{n} b_k$ is bounded if $x \neq 2 m \pi$.

(ii) $a_k = \frac{1}{k} \downarrow 0$

$\therefore$ Dirichlet's Test $\Rightarrow \sum_{k=1}^{\infty} \frac{\cos kx}{k} \in C, x \neq 2 m \pi$.

Exercise:

6.17: $\sum_{k=1}^{\infty} \frac{\sin kx}{k} \in C$, $\forall x$.

$[2 \sin \frac{1}{2} x \sin kx = \cos(k - \frac{1}{2})x - \cos(k + \frac{1}{2})x]$.

Example 6: $\sum_{k=1}^{\infty} (-1)^k \frac{k^k}{(k+1)^{k+1}} \in C$.

$0 \leq \frac{k^k}{(k+1)^{k+1}} = (\frac{k}{k+1})^k \frac{1}{k+1} \leq \frac{1}{k+1} \to 0$

Is the sequence monotone?

$a_n = \frac{n^n}{(n+1)^{n+1}} = (1 + \frac{1}{n})^{-n} \frac{1}{n+1}$

Now $(1 + \frac{1}{n})^n \uparrow e$ (Example 3, p. 51) so $(1 + \frac{1}{n})^{-n} \frac{1}{n+1} \downarrow 0$.

Exercises:

6.18:   In each case discuss whether $\sum\limits_{k=2}^{\infty} a_k$ is convergent where a_n is

the expression given

(i) $\dfrac{\sin nx}{\sqrt{\log n}}$

(ii) $(-1)^n \dfrac{2.4 \ldots (2n+2)}{1.3 \ldots (2n+1)}$

(iii) $(-1)^n \dfrac{1.3 \ldots (2n+1)}{2.4 \ldots (2n+2)}$

(iv) $\dfrac{(n!)^3}{(3n)!}$

(v) $\dfrac{1 + n + n^2}{n!}$

(vi) $\dfrac{1}{n^{1/n}}$

6.19:   Show

(i) $\sum\limits_{k=1}^{\infty} \left(\dfrac{1.3 \ldots 2k-1}{2.4 \ldots 2k} \right)^2 x^k \in \begin{cases} Abs\ C\ , & -1 < x < 1\ , \\ C\ , & x = -1\ , \\ D\ , & |x| > |\ \text{or}\ \ x = 1\ . \end{cases}$

(ii) $\sum\limits_{k=0}^{\infty} \dfrac{k^{100} x^k}{k!} \in Abs\ C\ ,\quad$ all x

(iii) $\sum\limits_{k=2}^{\infty} \dfrac{x^k}{(\log k)^2} \in \begin{cases} Abs\ C\ , & |x| < 1 \\ C\ , & x = -1 \\ D\ , & |x| > 1,\ x = 1 \end{cases}$

(iv) $\sum\limits_{k=1}^{\infty} (-1)^k \dfrac{x^{2k+1}}{2k+1} \in \begin{cases} Abs\ C\ , & |x| < 1 \\ C\ , & x = \pm 1 \\ D\ , & |x| > 1 \end{cases}$

(v) $\sum\limits_{n=0}^{\infty} \dfrac{x^n}{(1-x^{n+1})(1-x^{n+2})} = \begin{cases} \dfrac{1}{(1-x)^2}\ , & |x| < 1\ , \\ \dfrac{1}{x(1-x)^2}\ , & |x| > 1\ . \end{cases}$

[Show $S_n = \frac{1}{1-x} (\frac{1}{1-x} - \frac{x^{n+1}}{1-x^{n+2}})$, $x \neq \pm 1$]

(vi) $\sum\limits_{k=1}^{\infty} \frac{(2k+2)(2k+3) \ldots (3k+1)}{k!} \cdot \frac{x^{2k}}{2k+1} \in \begin{cases} \text{Abs } C , & x^2 \leq \frac{4}{27} , \\ \\ D , & x^2 > \frac{4}{27} . \end{cases}$

6.20: Suppose $a_n \downarrow 0$; show $0 \leq \sum\limits_{k=0}^{\infty} (-1)^k a_k \leq a_o$. Deduce that the

error made in truncating an alternating series of decreasing terms
is not greater than the absolute value of the first term neglected.

6.21: Prove that

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots = \log 2$

(ii) $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} \ldots = \frac{3}{2} \log 2$

(iii) $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} \ldots = 0$

(iv) $1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} + \frac{1}{7} - \frac{1}{10} \ldots = \frac{1}{2} \log 2$

[(i) From 6.20 deduce $|S_n(x) - \frac{1}{1+x}| \leq |x^n|$, $0 \leq x < 1$

and hence $|T_n(x) - \log(1 + x)| \leq |\frac{x^n}{n}|$, $0 \leq x \leq 1$

$S_n(x) = 1 - x + x^2 + \ldots + (-1)^{n-1} x^{n-1}$, $T_n(x) = \int_0^x S_n$.

(ii) Two positive terms followed by one negative.

(iii) One positive term followed by four negative.

(iv) Two positive terms followed by four negative.]

6.22: (i) Show that

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \quad + (-1)^{n+1} \frac{1}{\sqrt{n}} + \ldots\ldots$$

is convergent.

(ii) Show that

$$1 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{4}} + \ldots$$

is divergent.

6.23: (i) Show that the sum of an <u>absolutely</u> convergent series is not altered by rearranging the terms e.g. if $\sum\limits_{k=1}^{\infty} |a_k|$ is convergent then

$$a_2 + a_4 + a_1 + a_6 + a_8 + a_{10} + a_3 + \ldots$$

$$= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \ldots \quad .$$

(ii) A non-absolutely convergent series may be given <u>any</u> sum and may even be made to diverge simply by rearranging the terms. For this reason a series which is convergent but not absolutely convergent is called <u>conditionally</u> convergent.

6.24: Show that, if $m \neq 0$

$$\int_0^n \frac{dx}{m^2 + x^2} < \frac{1}{m^2} + \frac{1}{m^2 + 1} + \ldots + \frac{1}{m^2 + (n-1)^2} \leq \int_0^{n-1} \frac{dx}{m^2 + x^2} + \frac{1}{m^2}$$

and deduce that

$$\lim_{n \to \infty} \left\{ \frac{n}{n^2} + \frac{n}{1 + n^2} + \ldots + \frac{n}{(n-1)^2 + n^2} \right\} = \frac{\pi}{4}$$

6.25: Show

(i) $\lim\limits_{n \to \infty} \left\{ \dfrac{n^2}{1 + n^4} + \dfrac{2n^2}{2^4 + n^4} + \ldots + \dfrac{n^3}{2n^4} \right\} = \dfrac{\pi}{8}$

(ii) $\lim\limits_{n\to\infty} \{ \frac{1}{n} \log(1 + \frac{1}{n}) + \log(1 + \frac{2}{n}) + \ldots + \log 2 \} = 2 \log 2 - 1$

(iii) $\lim\limits_{n\to\infty} \frac{1}{n} \{ \frac{(2n)!}{n!} \}^{1/n} = \frac{4}{e}$

(iv) $\lim\limits_{n\to\infty} \{ (\frac{n}{n})^n + (\frac{n-1}{n})^n + \ldots + (\frac{1}{n})^n \} = \frac{e}{e-1}$

6.26: Prove that $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{(\log k)^q k^p}$ is (i) Abs C, $p > 1$, all q

(ii) C, $p > 0$, all q   (iii) D, $p < 0$, all q

(iv) C, $p = 0$, $q > 0$ (v) D, $p = 0$, $q \leq 0$.

6.27: Let $U = \sum\limits_{k=1}^{\infty} u_k$, $V = \sum\limits_{k=1}^{\infty} v_k$ be absolutely convergent and

$w_n = \sum\limits_{k=1}^{n} u_k v_{n-k}$; show that $\sum\limits_{k=1}^{\infty} w_k$ is absolutely convergent and

has sum UV.

[It has been shown that $(\sum\limits_{k=1}^{\infty} u_k)(\sum\limits_{k=1}^{\infty} v_k) = \sum\limits_{k=1}^{\infty} w_k$ if all three

series are convergent (Abel) and further it has also been shown

that if $\sum\limits_{k=1}^{\infty} u_k$ is convergent and $\sum\limits_{k=1}^{\infty} v_k$ is absolutely convergent

then $\sum\limits_{k=1}^{\infty} w_k$ is convergent and $(\sum\limits_{k=1}^{\infty} u_k)(\sum\limits_{k=1}^{\infty} v_k) = \sum\limits_{k=1}^{\infty} w_k$ (Mertens)]

6.28: Show

(i) $(\sum\limits_{k=0}^{\infty} x^k)(\sum\limits_{k=0}^{\infty} (k+1)x^k) = \sum\limits_{k=0}^{\infty} \frac{(k+1)(k+2)}{1.2} x^n$, if $|x| < 1$,

(ii) $\left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} \right) \left(\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \right) = \sum_{k=1}^{\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \ldots + \frac{1}{2k-1} \right) \frac{x^{2k}}{k}$,

if $|x| < 1$.

6.29: Show $\sum_{k=1}^{\infty} \frac{1}{k} \log\left(1 + \frac{1}{k}\right) \in C$.

6.30: Discuss the convergence or otherwise of

(i) $\sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right) \frac{\sin kx}{k}$,

(ii) $\sum_{k=1}^{\infty} (\sqrt{k+1} - \sqrt{k})(-1)^k$,

(iii) $\sum_{k=1}^{\infty} (\sqrt{k+1} - \sqrt{k}) \cos kx$.

(iv) $\sum_{k=1}^{\infty} \frac{b_k}{k}$, $\{b_n\} = \{2, -3, 2, -1, 2, -3, 2, -1, \ldots\}$

6.31: Suppose a_n is decreasing and $\sum_{k=1}^{\infty} a_k$ is convergent then

$\lim_{n \to \infty} n\, a_n = 0$.

6.32: (Cauchy Condensation Test) (i) If a_n is decreasing then

$$\sum_{k=1}^{\infty} a_k \in C \iff \sum_{k=1}^{\infty} 2^k a_{2^k} \in C$$

(ii) Use (i) to show $\sum_{k=1}^{\infty} \frac{1}{k^p} \in C$ if $p > 1$, $\in D$ if $p \leq 1$

(iii) Use (i) to investigate the convergence of

$$\sum_{k=2}^{\infty} \frac{1}{k(\log k)^p} , \quad \sum_{k=2}^{\infty} \frac{1}{k(\log k)(\log \log k)^p} . \quad [C , p > 1$$
$$D , p \leq 1] .$$

(iv) What other test could have been used in (ii), (iii) ?

6.33:   Prove or disprove

(i) $\sum\limits_{k=1}^{\infty} a_k \in C$, $a_n > 0 \Rightarrow \sum\limits_{k=1}^{\infty} (a_k\, a_{k+1})^{\frac{1}{2}} \in C$,

(ii) $\sum\limits_{k=1}^{\infty} (a_k\, a_{k+1})^{\frac{1}{2}} \in C$, $a_n \downarrow 0 \Rightarrow \sum\limits_{k=1}^{\infty} a_k \in C$.

6.34:   If $\sum\limits_{k=1}^{\infty} a_k \in Abs\ C$ then $\sum\limits_{k=1}^{\infty} a_k^{2}$, $\sum\limits_{k=1}^{\infty} \frac{a_k}{1+a_k}$ $(a_k \neq -1)$,

and $\sum\limits_{k=1}^{\infty} \frac{a_k^{2}}{1 + a_k^{2}} \in Abs\ C$.

6.35: (i) Prove $a_n > 0$, $\sum\limits_{k=1}^{\infty} a_k \in D \Rightarrow \sum\limits_{k=1}^{\infty} \frac{a_k}{1+a_k} \in D$, $\sum\limits_{k=1}^{\infty} \frac{a_k}{S_k} \in D$ and

$$\sum\limits_{k=1}^{\infty} \frac{a_k}{S_k^{2}} \in C \text{ where } S_n = \sum\limits_{k=1}^{n} a_k.$$

(ii) Prove $a_n > 0$, $\sum\limits_{k=1}^{\infty} a_k \in C \Rightarrow \sum\limits_{k=1}^{\infty} \frac{a_k}{1+a_k} \in C$, $\sum\limits_{k=1}^{\infty} \frac{a_k}{r_k} \in D$ and

$$\sum\limits_{k=1}^{\infty} \frac{a_k}{\sqrt{r_k}} \in C \text{ where } r_k = \sum\limits_{k=n}^{\infty} a_k.$$

6.36: (i) If $a_n > 0$ and $\sum\limits_{k=0}^{\infty} a_k \in D$ then $\exists\ \omega_n \downarrow 0 \ni \sum\limits_{k=0}^{\infty} \omega_k\, a_k \in D$.

(ii) If $a_n > 0$ and $\sum\limits_{k=0}^{\infty} a_k \in C$ then $\exists\ \omega_n \uparrow \infty \ni \sum\limits_{k=0}^{\infty} \omega_k\, a_k \in C$.

[This means there exists no 'best' comparison test for convergence or divergence.]

6.37:   Test for convergence or divergence

$$\sum_{k=2}^{\infty} (\log k)^{-\log k} \; , \; \sum_{k=3}^{\infty} (\log \log k)^{-\log \log k}$$

6.38:   We know $\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$. Show $\sum_{k\in S} \frac{1}{k} < 90,$ where S is the set of

natural numbers not containing the digit 0 in their decimal

representation e.g. $25 \in S$, $203 \notin S$.

6.39:   Does $\sum_{k=1}^{\infty} a_k \in C \Rightarrow \sum_{k=1}^{\infty} [|a_k|] \in C?$ $\sum_{k=1}^{\infty} |[a_k]|$?

[x] is the greatest integer not exceeding x .

6.40: $\sum_{k=2}^{\infty} \frac{1}{k \log k} = \infty$. How large must n be to ensure $\sum_{k=2}^{n} \frac{1}{k \log k} > 10$?.

IMPROPER INTEGRALS

Definition: We define

$$\int_a^{b-} f = \lim_{T \to b-} \int_a^T f \; , \quad \int_{a+}^b f = \lim_{T \to a+} \int_T^b f$$

$$\int_a^\infty f = \lim_{T \to \infty} \int_a^T f \; , \quad \int_{-\infty}^a f = \lim_{T \to -\infty} \int_T^a f$$

whenever these limits exist.

An improper integral is called <u>convergent</u> $(\epsilon\ C)$ if it exists and <u>divergent</u> $(\epsilon\ D)$ otherwise.

Examples:

(1) $\displaystyle\int_{0+}^1 \frac{1}{\sqrt{t}}\, dt = 2$, since $\displaystyle\int_T^1 \frac{1}{\sqrt{t}}\, dt = 2(1 - \sqrt{T})$, $T > 0$.

(2) $\displaystyle\int_{0+}^1 \frac{1}{t}\, dt \ \epsilon\ D$, since $\displaystyle\int_T^1 \frac{1}{t}\, dt = \log\left(\frac{1}{T}\right)$, $t > 0$.

(3) $\displaystyle\int_{0+}^1 \log t\, dt = -1$, since $\displaystyle\int_T^1 \log t\, dt = (t \log t - t)\Big|_T^1$

$$= -1 + T \log T - T \; , \quad T \quad 0 \; .$$

(4) $\displaystyle\int_0^\infty e^{-\alpha t}\, dt \begin{cases} = 1/\alpha \; , \; \alpha > 0 \\[2mm] \hspace{2cm} , \quad \text{since} \\[2mm] \epsilon\ D \; , \; \alpha \le 0 \end{cases}$

$$\int_0^T e^{-\alpha t}\, dt = \begin{cases} T \; , \; \alpha = 0 \\[3mm] \frac{1}{\alpha}(1 - e^{-\alpha T}) \; , \; \alpha \ne 0 \; . \end{cases}$$

Exercise:

6.41:   Show $\int_{1}^{\infty} t^p \, dt \in \begin{cases} C \;, \; p < -1 \\ \\ D \;, \; p \geq -1 \end{cases}$, $\int_{0+}^{1} t^p \, dt \in \begin{cases} C \;, \; p > -1 \\ \\ D \;, \; p \leq -1 \end{cases}$,

Show $\int_{0+}^{\infty} t^p \, dt \in D, \; \forall \; p$.

Theorem 6.7 (Cauchy Criterion): Suppose $\int_{0}^{T} f \; \exists \; \forall \; T > 0$;

then $\int_{0}^{\infty} f \in C \iff$ for each $\epsilon > 0 \; \exists \; T \ni$ if $T_2 \geq T_1 \geq T$ then $\left| \int_{T_1}^{T_2} f \right| < \epsilon$

Exercise:

6.42:   Write down the Cauchy Criterion for the convergence of $\int_{a+}^{b} f$.

Theorem 6.8 (Comparison Test):

Suppose $0 \leq f(t) \leq g(t)$ and $\int_{a}^{T} f$, $\int_{a}^{T} g \; \exists \; \forall \; T \geq a$ then

(i) $\int_{a}^{\infty} f \in D \Rightarrow \int_{a}^{\infty} g \in D$,

(ii) $\int_{a}^{\infty} g \in C \Rightarrow \int_{a}^{\infty} f \in C$.

A similar test applies to any improper integral.

Proof: $0 \leq \int_{a}^{T} f \leq \int_{a}^{T} g$ and both integrals are increasing functions of T.   □

Examples:

(1) (Gamma function)

$$\Gamma(p) = \int_{0+}^{\infty} e^{-t} \, t^{p-1} \, dt \in C \text{ if } p > 0$$

$$\int_{0+}^{\infty} e^{-t} \, t^{p-1} \, dt = \int_{0+}^{1} e^{-t} \, t^{p-1} \, dt + \int_{1}^{\infty} e^{-t} \, t^{p-1} \, dt$$

$$0 \le t \le 1 : 0 \le e^{-t} \, t^{p-1} \le t^{p-1} \quad \text{and} \quad \int_{0+}^{1} t^{p-1} \, dt = \frac{1}{p}, \quad p > 0$$

$$t \ge 1 : 0 \le e^{-t} \, t^{p-1} \le e^{-\frac{1}{2}t} \, e^{-\frac{1}{2}t} \, t^{p-1} \le C \, e^{-\frac{1}{2}t}$$

$$\text{and} \quad \int_{1}^{\infty} C \, e^{-\frac{1}{2}t} \, dt = 2 \, C \, e^{-\frac{1}{2}}, \quad \forall \, p \, .$$

(2) $\displaystyle\int_{1}^{\infty} \frac{|\sin x|}{x^2} \, dx \in C$

$\quad$ since $\displaystyle 0 \le \frac{|\sin x|}{x^2} \le \frac{1}{x^2}$ and $\displaystyle\int_{1}^{\infty} \frac{1}{x^2} \, dx \in C$

Exercise:

6.43: Show

(i) $\Gamma(p+1) = p \, \Gamma(p)$ $\quad$ [Integrate by parts]

(ii) $\Gamma(n+1) = n!$ $\quad n = 0, 1, 2, \ldots$ $\quad$ [Show $\Gamma(1) = 1$, use (i)]

(iii) $\displaystyle\int_{0+}^{\infty} e^{-at} \, t^{p-1} \, dt = \frac{1}{a^p} \, \Gamma(p)$ $\quad$ if $\quad a > 0$, $\quad p > 0$

(iv) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(\frac{3}{2}) = \frac{1}{2} \sqrt{\pi}$, $\Gamma(\frac{5}{2}) = \frac{1.3}{2.2} \sqrt{\pi}$

$\left[\Gamma(\frac{1}{2}) = \displaystyle\int_{0+}^{\infty} e^{-t} \, t^{-\frac{1}{2}} \, dt = 2 \int_{0}^{\infty} e^{-x^2} \, dx \right.$

$\therefore \; \Gamma(\frac{1}{2})^2 = 4 \left(\displaystyle\int_{0}^{\infty} e^{-x^2} \, dx \right) \left(\int_{0}^{\infty} e^{-y^2} \, dy \right)$

Consider $I_R^2 = (\int_0^R e^{-x^2} dx)(\int_0^R e^{-y^2} dy)$

$$= \int_0^R \int_0^R e^{-(x^2+y^2)} dx \, dy$$

From this deduce

$$\int_0^{\pi/2} \int_0^R e^{-r^2} r \, dr d\theta \leq I_R^2 \leq \int_0^{\pi/2} \int_0^{\sqrt{2}R} e^{-r^2} r \, dr \, d\theta$$

$$\therefore \quad \frac{\pi}{4} (1 - e^{-R^2}) \leq I_R^2 \leq \frac{\pi}{4} (1 - e^{-2R^2})$$

and hence $\Gamma(\frac{1}{2})^2 = \pi$.

(v) Given that $\Gamma(x)$ is known $1 \leq x < 2$ the relation (i) defines $\Gamma(x)$ elsewhere even for negative $x \neq 0, -1, -2, \ldots$. Use this to sketch the graph of $\Gamma(x)$. Check your work with Buck, p. 215.

Definition: $\int_0^\infty f \in \text{Abs } C$ if $\int_0^\infty |f| \in C$.

Theorem 6.9: $\int_0^\infty f \in \text{Abs } C \Rightarrow \int_0^\infty f \in C$

Proof: $|\int_{T_1}^{T_2} f| \leq \int_{T_1}^{T_2} |f|$, use Cauchy Criterion. $\qquad \square$

Exercises:

6.44: If $f \in C^1[0,\infty)$ and $\int_0^\infty |f'| \in C$ then show $\lim_{t \to \infty} f(t) \overset{\exists}{=} |$. Such a function is said to be of <u>Bounded Variation</u> (BV). You will encounter a slightly more general definition of BV in Math. 442 which is equivalent to f being the sum of two bounded increasing functions.

<u>Lemma 6.10.1</u> (Integration by parts formula): Suppose $f \in C^1[0,\infty)$, $g \in C[0,\infty)$,

and $G(t) = \displaystyle\int_0^t g$; then

$$\int_{t_1}^{t_2} f\, g = f\, G\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} f'\, G$$

<u>Theorem 6.10</u> (Dirichlet Test): $f \in C^1[0,\infty)$, $g \in C[0,\infty)$

If (i) $\left|\displaystyle\int_0^t g\right| \le B$, $\forall\, t \ge 0$,

   (ii) $\displaystyle\int_0^\infty |f'| \in C$ and $\lim_{t\to\infty} f(t) = 0$

then $\displaystyle\int_0^\infty f\, g \in C$.

[(ii) holds in particular if f is increasing or decreasing and has

limit 0].

Proof:

$$\left|\int_{t_1}^{t_2} f\, g\right| = \left|f\, G(t_2) - f\, G(t_1) - \int_{t_1}^{t_2} f'\, G\right|, \quad G(t) = \int_0^t g$$

$$\le \left\{|f(t_2)| + |f(t_1)| + \int_{t_1}^{t_2} |f'|\right\} B, \text{ by (i)},$$

$$\le \left\{\frac{\epsilon}{3B} + \frac{\epsilon}{3B} + \frac{\epsilon}{3B}\right\} B = \epsilon, \text{ by (ii)},$$

provided t_1, t_2 are sufficiently large.

<u>Example:</u> $\displaystyle\int_1^\infty \frac{\sin t}{t} \in C$

   (i) $\left|\displaystyle\int_1^T \sin t\, dt\right| = |1 - \cos T| \le 2$ (ii) $\frac{1}{t} \downarrow 0$ as $t \to \infty$

Exercises:

6.45:   Show $\int_1^\infty \frac{\sin t}{t} dt \notin Abs\ C$

6.46:   Show $\int_0^\infty \sin(t^2)dt \in C$, $\int_0^\infty \cos(e^t)dt \in C$

6.47   Show $\int_1^\infty \frac{\sin t}{t^p} \in \begin{cases} Abs\ C,\ p > 1 \\ C\ \ ,\ p > 0 \\ D\ \ ,\ p \le 0 \end{cases}$

Corollary 6.10.1 (Abel's Test): $f \in C^1[0,\infty)$, $g \in C[0,\infty)$

Suppose   (i) $\int_0^\infty g \in C$,

(ii) $\int_0^\infty |f'| \in C$.

Then $\int_0^\infty f\ g \in C$

[(ii) holds in particular if f is increasing or decreasing and bounded.]

Proof: Exercise 6.48

Exercises:

6.49:   Prove the following

(i) $\int_{0+}^1 \frac{dx}{(x+x^2)^{\frac{1}{2}}} \in C$,   (ii) $\int_{0+}^{1-} \frac{dx}{(x-x^2)^{\frac{1}{2}}} \in C$,

(iii) $\int_0^{1-} \frac{x\ dx}{1-x^2} \in D$,   (iv) $\int_{0+}^1 \frac{\log x}{\sqrt{x}} dx \in C$,

(v) $\displaystyle\int_1^\infty \frac{x^p}{1+x^q}\,dx \in$ Abs C $(q - p > 1)$; $\in D$ $(q - p \le 1)$.

(vi) $\displaystyle\int_1^\infty \frac{\sin(x^p)}{x}\,dx \in C\,(p > 0)$ but $\notin$ Abs C ; $\in D$ $(p = 0)$; $\in$ Abs C $(p < 0)$.

(vii) $\displaystyle\int_1^\infty \frac{1-\cos x}{x^q}\,dx \in$ Abs C $(q > 1)$; $\in D$ $(q \le 1)$.

6.50:   If f is uniformly continuous on $[0,\infty)$ show that $\displaystyle\int_0^\infty f \in C \Rightarrow \lim_{t\to\infty} f(t) = 0$

Show that this does not necessarily hold if 'uniformly' is omitted.

UNIFORM CONVERGENCE

Suppose $\lim\limits_{n \to \infty} f_n(x) = f(x)$, $a < x < b$. Under what circumstances

does $f_n \in C(a,b) \Rightarrow f \in C(a,b)$, $\lim\limits_{n \to \infty} \int_a^b f_n = \int_a^b f$, $\lim\limits_{n \to \infty} f_n' = f'$; equivalently

does $\sum\limits_{k=0}^{\infty} u_k(x) = u(x)$ imply $u \in C(a,b)$ if $u_k \in C(a,b)$, $\sum\limits_{k=0}^{\infty} \int_a^b u_k = \int_a^b u$,

$\sum\limits_{k=0}^{\infty} u_k' = u'$? The same questions for improper integrals:

If $\int_0^\infty f(x,t)dt = F(x)$ does $f(x,t)$ continuous in $x \Rightarrow F$ continuous, is

$$\int_0^\infty \{\int_a^b f(x,t)dx\}dt = \int_a^b \{\int_0^\infty f(x,t)dt\}dx = \int_a^b F , \int_0^\infty \frac{\partial}{\partial x} f(x,t)dt = F'(x) ?$$

Examples:

(1) $f_n(x) = \begin{cases} n(\frac{1}{n} - x), & 0 \le x \le \frac{1}{n} \\ \\ 0 & , \frac{1}{n} \le x \le 1 \end{cases}$

$f_n \in C[0,1]$

$\lim\limits_{n \to \infty} f_n(x) = f(x) = \begin{cases} 1 , & x = 0 \\ & \quad \therefore \quad f \notin C[0,1] \\ 0 , & 0 < x \le 1 \end{cases}$

(2) $f_n(x) = \begin{cases} 2n^2 x , & 0 \le x \le \frac{1}{2n} \\ -2n^2(x - \frac{1}{n}), & \frac{1}{2n} \le x \le \frac{1}{n} \\ 0 & , \frac{1}{n} \le x \le 1 \end{cases}$

$\therefore \lim\limits_{n \to \infty} f_n(x) = 0, 0 \le x \le 1$

$\int_0^1 (\lim\limits_{n \to \infty} f_n) = 0 \ne \frac{1}{2} = \lim\limits_{n \to \infty} \int_0^1 f_n$.

(3) The limit function might not even be Riemann integrable. Let

$\{r_1, r_2, r_3, \ldots\}$ be the rationals in $[0,1]$ and let $f_n(x) = 1$ if

$x = r_1, \ldots, r_n$, $f_n(x) = 0$ otherwise. Then $\lim\limits_{n \to \infty} f_n(x) = f(x)$ where

$f(x) = 1$ if x is rational and $f(x) = 0$ otherwise $\int_0^1 f_n = 0 \ \forall n$, but

$\int_0^1 f \neq$ (Exercise 3.15, p. 120)

Notice that in all three examples, given $\varepsilon > 0$, $x \in [0,1] \ \exists \ N \ni$ if

$n \geq N$, $|f_n(x) - f(x)| < \varepsilon$ (in fact $|f_n(x) - f(x)| = 0$). However

$N = N(\varepsilon, x)$ depends heavily on x.

Sequences and series of functions:

Suppose $D \subset R^m$ and $\{f_n\}$ is a sequence of real valued functions on D.

Definition:

(i) (Pointwise convergence) $\lim\limits_{n \to \infty} f_n = f$ on D if $\lim\limits_{n \to \infty} f_n(p) = f(p)$,

$\forall \ p \in D$; i.e. for each $\varepsilon > 0 \ \exists \ N = N(\varepsilon, p) \ni$ if $n \geq N$ then

$$|f(p) - f_n(p)| < \varepsilon.$$

(ii) (Uniform convergence) $\lim\limits_{n \to \infty} f_n = f$ <u>uniformly</u> on D if for each

$\varepsilon > 0 \ \exists \ N = N(\varepsilon) \ni$ if $n \geq N$ then

$$|f(p) - f_n(p)| < \varepsilon, \ \forall \ p \in D.$$

Theorem 6.11 (Cauchy Criterion):

$\{f_n\}$ is uniformly convergent on $D \iff$ for each $\varepsilon > 0 \ \exists \ N = N(\varepsilon) \ni$ if

$m, n \geq N$ then

$$|f_m(p) - f_n(p)| < \varepsilon, \ \forall \ p \in D$$

Proof: <u>Exercise</u> 6.51

<u>Proposition</u>:

(a)   (Negation of the definition of uniform convergence)

$\{f_n\}$ does <u>not</u> converge <u>uniformly</u> to f on $D \Longleftrightarrow \exists \ \epsilon_o > 0$, a

sequence of points $p_k \in D$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ $\ni$

$$|f_{n_k}(p_k) - f(p_k)| \geq \epsilon_o$$

(b)   (Negation of Cauchy criterion)

$\{f_n\}$ is not uniformly convergent on $D \Longleftrightarrow \exists \ \epsilon_o > 0$, and a sequence

of points $p_k \in D$ and $\forall \ k$, $\exists \ m, n \geq k \ni$

$$|f_n(p_k) - f_m(p_k)| \geq \epsilon_o \quad k = 1, 2, \ldots$$

<u>Examples</u>:

(4) $\lim\limits_{n \to \infty} \dfrac{x}{n} = 0$, $\forall \ x \in R$

$\lim\limits_{n \to \infty} \dfrac{x}{n} = 0$ uniformly on $[-M, M]$

since $\left|\dfrac{x}{n} - 0\right| = \dfrac{|x|}{n} \leq \dfrac{M}{N} < \epsilon$, if $n > \dfrac{M}{\epsilon}$, $\forall \ x \in [-M, M]$.

(5) $\lim\limits_{n \to \infty} \dfrac{x}{n} = 0$ nonuniformly on R

$f_n(x) = \dfrac{x}{n}$

$|f_n(n) - 0| = \dfrac{n}{n} = 1$, $n = 1, 2, \ldots$.

(6) $f_n(x) = \begin{cases} n\left(\dfrac{1}{n} - x\right) , & 0 \leq x \leq \dfrac{1}{n} \\ \\ 0 , & \dfrac{1}{n} \leq x \leq 1 \end{cases}$ (cf. Example 1)

$$\lim_{n \to \infty} f_n(x) = f(x) = \begin{cases} 1 \; , \; x = 0 \\ \\ 0 \; , \; 0 < x \le 1 \end{cases}$$

$\left| f_n(\frac{1}{2n}) - f(\frac{1}{2n}) \right| = \frac{1}{2}$, $\forall n$, so the convergence is nonuniform

on $[0,1]$. However, if $0 < \delta < 1$, $\{f_n\}$ is uniformly convergent on

$[\delta,1]$; if $n > \frac{1}{\delta}$ then $\frac{1}{n} < \delta$ and $\left| f_n(x) - 0 \right| = \left| 0 - 0 \right| < \epsilon$ if

$\epsilon > 0$ and $x \in [\delta,1]$.

(7) $\lim_{n \to \infty} e^{-nx} = \begin{cases} 1 \; , \; x = 0 \\ \\ 0 \; , \; x > 0 \end{cases} = f(x)$

The convergence is nonuniform on $[0,\infty)$; $f_n(x) = e^{-nx}$,

$$\left| f_n(\frac{1}{n}) - f(\frac{1}{n}) \right| = \left| e^{-1} - 0 \right| = \frac{1}{e}$$

The convergence is uniform on $[\delta,\infty)$ if $\delta > 0$

since $\left| e^{-nx} - 0 \right| = e^{-nx} \le e^{-n\delta}$ if $x \ge \delta$ and $\lim_{n \to \infty} e^{-n\delta} = 0$.

Exercises:

6.52:   Let $g_n(x) = \begin{cases} nx \; , \; 0 \le x \le \frac{1}{n} \\ \\ \dfrac{1}{nx} \; , \; \dfrac{1}{n} \le x \end{cases}$

(i)   Sketch the graphs of a few of the functions g_n .

(ii)   Find $\lim_{n \to \infty} g_n(x)$.

(iii)   Prove the convergence is nonuniform on $[0,\infty)$ and on $(0,\infty)$.

(iv)   Prove the convergence is uniform on $[\delta,\infty)$ if $\delta > 0$.

6.53: $f_n(x) = \begin{cases} nx \; , \; 0 \le x \le \frac{1}{n} \\ \\ \dfrac{n}{n-1}(1-x), \; \dfrac{1}{n} < x \le 1 \end{cases}$

(i)   Find $\lim_{n \to \infty} f_n(x)$ $0 \le x \le 1$

(ii) Show that the convergence is uniform on $[\delta, 1]$ if $\delta > 0$ but is nonuniform on $(0,1]$ and on $[0,1]$.

Theorem 6.12:   Suppose

(i) f_n is continuous on D, $\forall n$,

(ii) $f_n \to f$ uniformly on D .

   Then f is continuous on D .

Proof:

   Given $\epsilon > 0$, choose $N \ni |f_N(p) - f(p)| < \frac{\epsilon}{3} \; \forall \; p \in D$ (by (ii)) .

$$|f(p) - f(p_o)| = |f(p) - f_N(p) + f_N(p) - f_N(p_o) + f_N(p_o) - f(p_o)|$$

$$\le |f(p) - f_N(p)| + |f_N(p) - f_N(p_o)| + |f_N(p_o) - f(p_o)|$$

$$< \frac{2\epsilon}{3} + |f_N(p) - f_N(p_o)|$$

   But f_N is continuous on D   (by (i)) so $\exists \; \delta > 0$

$$\ni |f_N(p) - f_N(p_o)| < \frac{\epsilon}{3} \text{ if } |p - p_o| < \delta \; .$$

   $\therefore$ if $|p - p_o| < \delta$, then $|f(p) - f(p_o)| < \epsilon$. $\square$

Theorem 6.13:   Suppose

(i) $D \subset R^m$ has Jordan content and is compact,

(ii) $f_n : D \to R$ is continuous on D, $n = 1, 2, \ldots$,

(iii) $f_n \to f$ uniformly on D

Then

$$\lim_{n \to \infty} \int_D f_n = \int_D f = \int_D (\lim_{n \to \infty} f)$$

Proof: $\int_D f_n$, $\int_D f \; \exists$ by Theorems 6.12, 3.4.

Given $\epsilon > 0$, $\exists$ N $\ni$ if $n \geq N$ then

$$|f_n(p) - f(p)| < \epsilon , \; \forall \; p \in D$$

$$\therefore \; \left| \int_D f_n - \int_D f \right| = \left| \int_D f_n - f \right|$$

$$\leq \int_D |f_n - f| < \epsilon \, \mu(D), \text{ if } n \geq N.$$

i.e. $\lim_{n \to \infty} \int_D f_n = \int_D f$ □

Corollary 6.13.1: $f_n : [a,b] \to R$. Suppose

    (i) $f_n \in C^1[a,b]$, $n = 1, 2, \ldots$,

    (ii) $\{f_n(x_0)\}$ is convergent for some $x_0 \in [a,b]$,

    (iii) $\{f_n'\}$ is uniformly convergent on $[a,b]$.

Then $\exists$ f $: [a,b] \to R \ni f \in C^1[a,b]$, $f_n \to f$ and $f_n' \to f'$ both

uniformly on $[a,b]$.

Proof: $f_n(x) = f_n(x_0) + \int_{x_0}^x f_n'$, $\forall \; x \in [a,b]$ (*)

Given $\epsilon > 0$, $\exists$ N $\ni$ if m, n $\geq$ N then $|f_n'(x) - f_m'(x)| < \epsilon, \; \forall \; x \in [a,b]$ (by (iii))

and $|f_n(x_0) - f_m(x_0)| < \epsilon$ (by (ii))

$$\therefore \quad |f_n(x) - f_m(x)| \le |f_n(x_o) - f_m(x_o)| + \left| \int_{x_o}^{x} (f_n' - f_m') \right|$$

$$< \epsilon + \epsilon |x - x_o|$$

$$\le \epsilon[1 + (b-a)] \ , \quad \forall \ x \ \epsilon[a,b] \ .$$

$\therefore \quad \{f_n\}$ is also uniformly convergent on $[a,b]$.

$$\text{Let} \quad f = \lim_{n \to \infty} f_n \ , \quad g = \lim_{n \to \infty} f_n' \ ;$$

$g \ \epsilon \ C[a,b]$, by Theorem 6.12, and from (*)

$$f(x) = f(x_o) + \int_{x_o}^{x} g \ , \quad \forall \ x \ \epsilon[a,b], \quad \text{by Theorem 6.13}$$

so $\quad f'(x) = g(x) \ , \quad \forall \ x \ \epsilon[a,b] \ .$ □

Remark: Corollary 6.13.1 is valid also if the assumption that "$f_n \ \epsilon \ C^1[a,b]$" is relaxed to "$f_n' \ \rightrightarrows$| on $[a,b]$"; the present proof is no good then however (why?). For a proof of this more general result see Bartle p. 217.

We summarize these results as they pertain to infinite series.

Definition: $\sum_{k=1}^{\infty} u_k(p) = S(p)$ is uniformly convergent on D (ϵ Unif $C(D)$)

if $S_n \to S$ uniformly on D where $S_n(p) = \sum_{k=1}^{n} u_k(p)$.

Theorem 6.12': If each $u_k(p)$ is continuous on D and $\sum_{k=1}^{\infty} u_k \ \epsilon$ Unif $C(D)$ then $\sum_{k=1}^{\infty} u_k$ is continuous on D.

Theorem 6.13': If D has content, u_k continuous on D, $\sum_{k=1}^{\infty} u_k \ \epsilon$ Unif $C(D)$ then

$$\sum_{k=1}^{\infty} \left(\int_{D} u_k \right) = \int_{D} \left(\sum_{k=1}^{\infty} u_k \right)$$

<u>Corollary 6.13.1:</u> If $u_k \in C^1[a,b]$, $\sum\limits_{k=1}^{\infty} u_k(x_0) \in C$,

$$\sum_{k=1}^{\infty} u_k' \in \text{Unif } C([a,b]) \text{ then}$$

$$\sum_{k=1}^{\infty} \frac{d}{dx} u_k(x) = \frac{d}{dx} \sum_{k=1}^{\infty} u_k(x) \qquad \forall \, x \in [a,b]$$

<u>Theorem 6.14</u> (Weierstrass Comparison Test):   Suppose

(i) $|u_k(p)| \leq M_k$, $\forall \, p \in D$

(ii) $\sum\limits_{k=1}^{\infty} M_k \in C$

Then $\sum\limits_{k=1}^{\infty} u_k(p)$ is absolutely and uniformly convergent on D.

Proof:

$$\left| \sum_{k=n}^{m} u_k(p) \right| \leq \sum_{k=n}^{m} |u_k(p)| \leq \sum_{k=n}^{m} M_k, \forall \, p \in D \text{ ; use the Cauchy criterion.}$$

□

Examples:

(8) $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$, $|x| < 1$

If $|x| \leq r$, $|(-1)^n x^{2n}| \leq r^{2n}$ and $\sum\limits_{k=0}^{\infty} r^{2k} \in C$, if $r < 1$

$\therefore \dfrac{1}{1+x^2} = \sum\limits_{k=0}^{\infty} (-1)^k x^{2k}$ is absolutely and uniformly convergent

$-r \leq x \leq r$ if $0 \leq r < 1$.

$\therefore$ Integration from 0 to x, $|x| < 1$, we find

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

$$\therefore \ \left| \tan^{-1} x - \sum_{k=0}^{N} (-1)^k \frac{x^{2k+1}}{2k+1} \right| \le \frac{1}{2N+3} , \quad |x| < 1$$

(cf. Exercise 6.20). Since the expression on the left is continuous

on [0,1] we take the limit $x \to 1-$ and obtain

$$\left| \frac{\pi}{4} - \sum_{k=0}^{N} (-1)^k \frac{1}{2k+1} \right| \le \frac{1}{2N+3}$$

$$\therefore \ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots$$

Question: Is this a good expression for approximating π? How many terms
in the series must be taken to approximate $\frac{\pi}{4}$ to two decimals?

(9) $$\int_0^1 [\sum_{k=1}^{\infty} \frac{x}{k(x+k)}] dx = \gamma \ \text{(Euler's constant, Exercise 6.12)}$$

$$\left| \frac{x}{k(x+k)} \right| \le \frac{1}{k^2} , \quad 0 \le x \le 1 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^2} \in C$$

$$\therefore \ \sum_{k=1}^{\infty} \frac{x}{k(x+k)} \equiv F(x) \ \text{is uniformly convergent on [0,1] by the}$$

Weierstrass Test.

Theorem 6.12 => $F \in C[0,1]$ and Theorem 6.13 =>

$$\int_0^1 F = \int_0^1 [\sum_{k=1}^{\infty} \frac{x}{k(x+k)}] dx = \sum_{k=1}^{\infty} [\int_0^1 \frac{x}{k(x+k)} \ dx]$$

$$= \sum_{k=1}^{\infty} \int_0^1 [\frac{1}{k} - \frac{1}{x+k}] dx = \sum_{k=1}^{\infty} [\frac{1}{k} - \log (\frac{k+1}{k})]$$

$$= \lim_{N \to \infty} \sum_{k=1}^{N} [\frac{1}{k} - \log (\frac{k+1}{k})]$$

$$= \lim_{N \to \infty} [1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N} - \log(N+1)]$$

$$= \lim_{N \to \infty} [1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N} - \log N - \log(\frac{N+1}{N})] = \gamma .$$

Exercises:

6.54: (i) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$ is uniformly convergent if $|x| \le r$, $r < 1$.

(ii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ if $|x| < 1$

(iii) $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

6.55: $\sum\limits_{k=1}^{\infty} \frac{\sin kx}{k^p}$ is absolutely and uniformly convergent, $-\infty < x < \infty$, if $p > 1$.

[Weierstrass Test].

6.56: Find a sequence of functions f_n such that $\lim\limits_{n \to \infty} \int_0^1 f_n = 0$ but $\{f_n(x)\}$

is not convergent for any $x \in [0,1]$.

6.57 (Wallis' Formula):

(i) Show $\lim\limits_{n \to \infty} \int_0^{\pi/2} (\sin x)^n dx = 0$ even though $(\sin x)^n$ is not uniformly

convergent on $[0, \frac{\pi}{2}]$.

(ii) If $S_n = \int_0^{\pi/2} (\sin x)^n dx$ then $S_n = \frac{n-1}{n} S_{n-2}$, $n \ge 2$.

(iii) $S_{2n} = \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots (2n)} \frac{\pi}{2}$

$S_{2n+1} = \frac{2.4.6 \ldots (2n)}{1.3.5 \ldots (2n+1)}$

(iv) $\{S_n\}$ is a decreasing sequence.

(v) If $w_n = \frac{2.2.4.4.6.6 \ldots (2n)(2n)}{1.3.3.5.5.7 \ldots (2n-1)(2n+1)}$ then

$\lim\limits_{n \to \infty} w_n = \frac{\pi}{2}$.

(vi) $\lim\limits_{n \to \infty} \left[\frac{(n!)^2 \, 2^{2n}}{(2n)! \, \sqrt{n}} \right] = \sqrt{\pi}$

(vii) Think about Exercise 6.57(i). Can you relax the condition in Theorem 6.13 which requires $\{f_n\}$ to be uniformly convergent on D in order that $\lim\limits_{n\to\infty} \int_D f_n = \int_D (\lim\limits_{n\to\infty} f_n)$?

Tests for nonabsolute uniform convergence:

Theorem 6.15 (Dirichlet's Test): Suppose

(i) $\left| \sum\limits_{k=1}^{n} v_k(p) \right| \leq B$, $\forall\, p \in D$, $n = 1, 2, \ldots$,

(ii) $u_n(p) \to 0$ for each $p \in D$ and $\sum\limits_{k=1}^{\infty} |u_{k+1}(p) - u_k(p)|$ is uniformly convergent on D.

Then $\sum\limits_{k=1}^{\infty} u_k(p)\, v_k(p)$ is uniformly convergent on D.

[(ii) holds in particular if $u_n \downarrow 0$ $(n \to \infty)$ uniformly on D] .

Proof: Condition (ii) implies $u_n(p) \to 0$ uniformly on D since

$$\left| u_m(p) - u_n(p) \right| = \left| \sum_{k=n}^{m-1} u_{k+1}(p) - u_k(p) \right|$$

$$\leq \sum_{k=n}^{m} |u_{k+1}(p) - u_k(p)| \quad \text{(use the Cauchy Criterion)}.$$

Now, given $\varepsilon > 0$, (ii) $\Longrightarrow \exists\, N \ni$ if $n, m \geq N$ then

$$|u_n(p)| < \frac{\varepsilon}{3B} , \quad \sum_{k=n}^{m} |u_{k+1}(p) - u_k(p)| < \frac{\varepsilon}{3B} , \quad \forall\, p \in D .$$

$$\therefore \quad \left| \sum_{k=n}^{m} u_k(p)\, v_k(p) \right| = \left| \sum_{k=n}^{m} u_k(p) \{ S_k(p) - S_{k-1}(p) \} \right| , \quad S_k(p) = \sum_{j=1}^{k} v_j(p) ,$$

$$= \left| u_{m+1}(p)\, S_m(p) - u_n(p)\, S_{n-1}(p) - \sum_{k=n}^{m} S_k(p) \{ u_{k+1}(p) - u_k(p) \} \right|$$

$$\leq B\{|u_{m+1}(p)| + |u_n(p)| + \sum_{k=n}^{m} |u_{k+1}(p) - u_k(p)|$$

$$< B\{\frac{\epsilon}{3B} + \frac{\epsilon}{3B} + \frac{\epsilon}{3B}\} = \epsilon , \forall p \in D , \text{ by (i).} \qquad \square$$

Example:

$\sum_{k=1}^{\infty} \frac{\sin kx}{k^{\alpha}}$ is absolutely and uniformly convergent, $-\infty < x < \infty$, if $\alpha > 1$,

by the Weierstrass Test.

$\sum_{k=1}^{\infty} \frac{\sin kx}{k^{\alpha}}$ is uniformly convergent, $2m\pi + \delta \leq x \leq (2m+1)\pi - \delta$, $\delta > 0$,

if $0 < \alpha \leq 1$, since:

$$\text{(i)} \quad |\sum_{k=1}^{n} \sin kx| = \left| \frac{\cos \frac{1}{2} x - \cos (n + \frac{1}{2})x}{2 \sin \frac{1}{2} x} \right|$$

$$\leq \frac{1}{\sin \frac{1}{2} \delta} , \quad 2m\pi + \delta \leq x \leq (2m + 1)\pi - \delta$$

(ii) $\frac{1}{k^{\alpha}} \downarrow 0$ (uniformly in x since x absent)

Theorem 6.16 (Abel's Test): Suppose

(i) $\sum_{k=1}^{\infty} v_k(p)$ is uniformly convergent on D,

(ii) $|u_n(p)| \leq B$, $\sum_{k=1}^{\infty} |u_{k+1}(p) - u_k(p)| \leq B$, $\forall p \in D$.

Then $\sum_{k=1}^{\infty} u_k(p) v_k(p)$ is uniformly convergent on D.

[(ii) holds in particular if, for each $p \in D$, $u_n(p)$ is either increasing or

decreasing in n and $|u_n(p)| \leq B$. <u>Please note</u> $\{u_n(p)\}$ need not be uniformly

convergent; the bound B must be uniform.]

Proof: If $\epsilon > 0$, $\exists$ N $\ni$ if $m, n \geq N$ and $S_k(p) = \sum\limits_{j=n}^{k} v_j(p)$ (N.B.)

then $|S_m(p)| \leq \frac{\epsilon}{2B}$, $\forall$ $p \in D$.

$\therefore$ $|\sum\limits_{k=n}^{m} u_k(p) v_k(p)| = |\sum\limits_{k=n}^{m} u_k(p) \{ S_k(p) - S_{k-1}(p) \}|$

$= |u_{m+1}(p) S_m(p) - u_n(p) S_{n-1}(p) + \sum\limits_{k=n}^{m} S_k(p) \{u_{k+1}(p) - u_k(p)\}|$

$\leq |u_{m+1}(p)||S_m(p)| + \sum\limits_{k=n}^{m} |S_k(p)||u_{k+1}(p) - u_k(p)|$, $S_{n-1}(p) = 0$,

$\leq B \{\frac{\epsilon}{2B} + \frac{\epsilon}{2B}\} = \epsilon$, since $|S_k(p)| \leq \frac{\epsilon}{2B}$. $\qquad\qquad$ $\square$

Example:

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \ldots , \quad |x| < 1$$

This is uniformly convergent $|x| \leq r < 1$.

$$\therefore \quad \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \qquad (*)$$

Weierstrass Test $\Rightarrow$ uniform convergence $|x| \leq r < 1$. However this series is also convergent for $x = \pm 1$ (e.g. Leibniz Test). Is it in fact uniformly convergent $-1 \leq x \leq 1$? Yes.

(i) $v_n(x) = \frac{(-1)^n}{2n+1}$

$\sum\limits_{k=0}^{\infty} v_k(x)$ is uniformly convergent $-1 \leq x \leq 1$ since it is convergent

and x is absent.

(ii) $u_n(x) = x^{2n+1}$

$|u_n(x)| \leq 1$ and $\{u_n(x)\}$ is a monotone sequence for each x.

∴ by Abel's Test the series (*) is uniformly convergent $|x| \leq 1$.

Thus both the left and right-hand sides of (*) are continuous functions on $[-1,1]$ and we have (again)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Power Series:

The series $\sum\limits_{k=0}^{\infty} a_k x^k$, $\sum\limits_{k=0}^{\infty} a_k (x - c)^k$ are power series in

x and $(x-c)$ respectively if a_k are constants.

Theorem 6.17 (Radius of convergence) $\exists$ R, $0 \leq R \leq \infty$ ∋

(i) $\sum\limits_{k=0}^{\infty} a_k x^k \in Abs\ C$, $|x| < R$,

(ii) $\sum\limits_{k=0}^{\infty} a_k x^k \in D$, $|x| > R$,

(iii) $\sum\limits_{k=0}^{\infty} a_k x^k \in ?$, $x = \pm R$.

R is called the radius of convergence of $\sum\limits_{k=0}^{\infty} a_k x^k$.

Proof: If $\sum\limits_{k=0}^{\infty} a_k x_o^k \in C$ then $\sum\limits_{k=0}^{\infty} a_k x^k \in Abs\ C$ if $|x| < |x_o|$

since $|a_n x^n| = |a_n x_o^n| \left|\frac{x}{x_o}\right|^n \leq M \left|\frac{x}{x_o}\right|^n$, for all n if $|x| < |x_o|$ and

$\sum\limits_{k=0}^{\infty} a_k x_o^k \in C$;

but $M \sum\limits_{k=0}^{\infty} \left|\frac{x}{x_o}\right|^k \in C$, if $|x| < |x_o|$ ∴ $\sum\limits_{k=0}^{\infty} a_k x^k \in Abs\ C$, if $|x| < |x_o|$

(Comparison Test) □

Examples:

(1) $\displaystyle\sum_{k=0}^{\infty} x^k \in \begin{cases} Abs\ C\ ,\ |x| < 1 \\ \\ D\ \ \ \ ,\ |x| \geq 1 \end{cases}$, $R = 1$

(2) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k} \in \begin{cases} Abs\ C\ ,\ |x| < 1 \\ C\ \ \ \ ,\ x = -1 \\ D\ \ \ \ ,\ x = 1,\ |x| > 1 \end{cases}$, $R = 1$

(3) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^2} \in \begin{cases} Abs\ C\ ,\ |x| \leq 1 \\ \\ D\ \ \ \ ,\ |x| > 1 \end{cases}$, $R = 1$

(4) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k!} \in Abs\ C\ ,\ \ all\ x$, $R = \infty$

(5) $\displaystyle\sum_{k=1}^{\infty} k!\ x^k \in \begin{cases} Abs\ C,\ x = 0 \\ \\ D\ \ \ ,\ x \neq 0 \end{cases}$, $R = 0$

Corollary 6.17.1: Let R be as in Theorem 6.17. If $0 \leq R_1 < R$ then $\displaystyle\sum_{k=0}^{\infty} a_k x^k$ is uniformly convergent for $|x| \leq R_1$.

Proof: Theorem 6.17 $\Rightarrow \displaystyle\sum_{k=0}^{\infty} a_k R_1{}^k \in Abs\ C$

i.e. $\displaystyle\sum_{k=0}^{\infty} |a_k| R_1{}^k \in C$.

If $|x| \leq R_1$ then $|a_n x^n| \leq |a_n| R_1{}^n \Rightarrow \displaystyle\sum_{k=0}^{\infty} a_k x^k \in Unif\ C(|x| \leq R_1)$

(Weierstrass Test) □

<u>Theorem 6.18:</u>

(i) $\sum\limits_{k=1}^{\infty} k \, a_k \, x^{k-1}$ and (ii) $\sum\limits_{k=0}^{\infty} \dfrac{a_k \, x^{k+1}}{k+1}$

both have the same radius of convergence as $\sum\limits_{k=0}^{\infty} a_k \, x^k$.

Proof:

$$\left| \frac{(n+1) \, a_{n+1} \, x^n}{n \, a_n \, x^{n-1}} \right| = \left(1 + \frac{1}{n}\right) \left| \frac{a_{n+1}}{a_n} \right| |x|$$

$$< \left| \frac{a_{n+1}}{a_n} \right| |x_o| \, , \text{ if } |x| < |x_o| \text{ and } n \text{ large enough,}$$

$$= \left| \frac{a_{n+1} \, x_o^{n+1}}{a_n \, x_o^n} \right|$$

$\therefore \sum\limits_{k=0}^{\infty} a_k \, x_o^k \in \text{Abs } C \Rightarrow \sum\limits_{k=1}^{\infty} k \, a_k \, x^{k-1} \in \text{Abs } C, \text{ if } |x| < |x_o| \, , \text{ by the Ratio}$

Comparison Test.

Similarly

$$\left| \frac{a_{n+1} \, x^{n+1}}{a_n \, x^n} \right| = \left| \frac{a_{n+1}}{a_n} \right| |x| < \left(1 + \frac{1}{n}\right) \left| \frac{a_{n+1}}{a_n} \right| |x|$$

$$= \left| \frac{(n+1) \, a_{n+1} \, x^n}{n \, a_n \, x^{n-1}} \right|$$

so $\sum\limits_{k=1}^{\infty} k \, a_k \, x^{k-1} \in \text{Abs } C \Rightarrow \sum\limits_{k=0}^{\infty} a_k \, x^k \in \text{Abs } C$.

This proves the claim about (i); note that we have also proved the claim about (ii). $\square$

Corollary 6.18.1 (N.B.):

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $|x| < R$, $R > 0$

then $f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$, $|x| < R$,

and $\int_0^x f = \sum_{k=0}^{\infty} \frac{a_k x^{k+1}}{k+1}$, $|x| < R$,

Proof: All the series are uniformly convergent $|x| \leq R_1 < R$ so we may use

Theorems 6.12, 6.13 and Corollary 6.13.1. $\square$

Corollary 6.18.2:

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $|x| < R$, $R > 0$ then f has derivatives

of all orders on $(-R, R)$.

Exercises:

6.58: (i) $\sum_{k=0}^{\infty} a_k x^k = 0$, $\forall |x| < R$, $\Rightarrow a_n = 0 \ \forall n$ if $R > 0$.

(ii) $\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} b_k x_k$, $\forall |x| < R \Rightarrow a_n = b_n \ \forall n$ if $R > 0$.

6.59: Prove $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$, $\forall x$

and deduce $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots$, $\forall x$

6.60: Given that the equation $y'' + y = 0$ has a solution of the form

$y = \sum_{k=0}^{\infty} a_k x^k$, prove that $y = c_1(x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots) + c_2(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots)$

where c_1 and c_2 are arbitrary constants.

Theorem 6.19 (Abel's Theorem): Suppose

(i) $\sum\limits_{k=0}^{\infty} a_k x^k \in C$, $|x| < R$,

(ii) $\sum\limits_{k=0}^{\infty} a_k R^k \in C$.

Then $\sum\limits_{k=0}^{\infty} a_k x^k$ is uniformly convergent $0 \le x \le R$

[Compare with Corollary 6.17.1.]

Proof: The proof uses Abel's Test (Theorem 6.16)

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k R^k \left(\frac{x}{R} \right)^k$$

(i) $\sum\limits_{k=0}^{\infty} a_k R^k$ uniformly convergent $0 \le x \le R$ (x is absent)

(ii) $\left(\frac{x}{R} \right)^n$ is decreasing and $0 \le \left(\frac{x}{R} \right)^n \le 1$, $0 \le x \le R$

$\therefore \sum\limits_{k=0}^{\infty} a_k x^k$ is uniformly convergent $0 \le x \le R$.   □

Exercise:

6.61:   Show $\sum\limits_{k=0}^{\infty} a_k R^k \in C \Rightarrow \sum\limits_{k=0}^{\infty} a_k x^k$ is uniformly convergent

$-R + \delta \le x \le R$, $\delta > 0$.

Corollary 6.19.1:

$$\sum_{k=0}^{\infty} a_k R^k \in C \Rightarrow \lim_{x \to R-} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k R^k$$

Proof: By Theorem 6.19 and Theorem 6.12, $\sum\limits_{k=0}^{\infty} a_k x^k$ is continuous, $0 \le x \le R$. □

<u>Examples:</u>

(6)   We have seen $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$, $|x| < 1$

(integrate $\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \dots$, $|x| < 1$). Leibniz's

Test shows the series is also convergent for $x = \pm 1$. From

Corollary 6.19.1 we deduce

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

[Compare with the proof we gave using the Weierstrass Test.]

(7) $\frac{1}{1 + x^8} = 1 - x^8 + x^{16} - x^{24} + x^{32} \dots$ $|x| < 1$

$\int_0^x \frac{1}{1 + x^8} \, dx = x - \frac{x^9}{9} + \frac{x^{17}}{17} - \frac{x^{25}}{25} + \dots$ $|x| < 1$

This also converges for $x = \pm 1$ so it is uniformly convergent

$|x| \leq 1$. Take $\lim\limits_{x \to 1-}$:

$$\int_0^1 \frac{1}{1 + x^8} \, dx = 1 - \frac{1}{9} + \frac{1}{17} - \frac{1}{25} + \frac{1}{33} \dots$$

In particular $\left| \int_0^1 \frac{1}{1 + x^8} \, dx - 1 + \frac{1}{9} - \frac{1}{17} + \frac{1}{25} \right| \leq \frac{1}{33}$ (why?)

<u>Exercises:</u>

6.62: (i) Show $(1+x)^\alpha = 1 + \frac{\alpha}{1!} x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \dots$, $|x| < 1$,

(ii) Deduce $\sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{1 \cdot 2} \cdot \frac{1}{2^2} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2^3} \cdot \frac{x^7}{7} + \dots$, $|x| < 1$,

(iii) Deduce $\frac{\pi}{2} = 1 + \frac{1}{2} \cdot \frac{1}{3} + \frac{1 \cdot 3}{1 \cdot 2} \cdot \frac{1}{2^2} \cdot \frac{1}{5} + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2^3} \cdot \frac{1}{7} + \dots$.

6.63: Show that the converse of Corollary 6.19.1 is not true

i.e. $\lim\limits_{x \to R-} \sum\limits_{k=0}^{\infty} a_k x^k \ni \not\Rightarrow \sum\limits_{k=0}^{\infty} a_k R^k \in C$

[Consider the expansion in powers of x of $\frac{1}{1+x}$ near $x = 1$.]

6.64: Let $S_n(x) = \frac{1}{x^2} + \frac{1}{n} e^{-nx}$, $0 < x \le 1$; does $\{S_n\}$ converge uniformly

on $(0,1]$?

6.65: Show that $\sum\limits_{k=0}^{\infty} x^k$ converges uniformly on $[-a,a]$ if $0 < a < 1$,

but not on $(-1,1)$.

6.66: Show that $\sum\limits_{k=0}^{\infty} \frac{x^k}{k!}$ converges uniformly on every interval $[-a,a]$

but not on $(-\infty,\infty)$.

6.67: Let $\{a_n\}$ be a sequence of constants such that $\sum\limits_{k=0}^{\infty} a_k \in$ Abs C ;

show that $\sum\limits_{k=0}^{\infty} a_k \cos kx$, $\sum\limits_{k=1}^{\infty} a_k \sin kx$ both converge uniformly

on $(-\infty,\infty)$.

6.68: Discuss the convergence or otherwise (absolute? uniform?
conditional?) of $\sum\limits_{k=0}^{\infty} (x \log x)^k$, $x > 0$.

6.69: Let $f(x) = \sum\limits_{k=1}^{\infty} k e^{-kx}$. Where is f continuous? Show that

$\int_1^2 f = \frac{e}{e^2-1}$. Justify each step of your work.

6.70: Let $u_n(x)$ be a sequence of positive nondecreasing continuous functions on $[a,b]$. Prove that

$$\int_a^b [\sum_{k=0}^{\infty} u_k(x)]dx = \sum_{k=0}^{\infty} [\int_a^b u_k(x)dx]$$

if $\sum_{k=0}^{\infty} u_k(b)$ is convergent.

6.71: If $f(x) = \sum_{k=1}^{\infty} a_k x^k$ converges for $x = x_o \neq 0$ show that

$$f'(x) = \sum_{k=1}^{\infty} k\, a_k\, x^{k-1} \text{ if } |x| < |x_o| .$$

6.72: Show that if $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k + x^2}$ then

$$\int_0^x 2\, f(t)t\, dt = \sum_{k=1}^{\infty} (-1)^k \log(1 + \frac{x^2}{k}) . \text{ Justify your work.}$$

6.73: Justify the equation

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{m + k\alpha} = \int_0^1 \frac{t^{m-1}}{1 + t^{\alpha}}\, dt , \quad m \geq 1 , \quad \alpha > 0 .$$

Uniform convergence of improper integrals:

Definition: $\int_0^{\infty} f(p,t)dt = F(p)$, $p \in D$, is uniformly convergent on D if

$$\lim_{T \to \infty} \int_0^T f(p,t) = F(p) \text{ uniformly on } D \text{ i.e. for each } \varepsilon > 0, \exists R = R(\varepsilon) \ni$$

if $T \geq R$. Then $\left| \int_0^T f(p,t)dt - F(p) \right| < \varepsilon , \forall p \in D$.

Theorem 6.20 (Cauchy Criterion): $\int_0^{\infty} f(p,t)dt$ is uniformly convergent on

$D \Longleftrightarrow$ for each $\varepsilon > 0 \exists R \ni$ if $T_2 > T_1 \geq R$ then $\left| \int_{T_1}^{T_2} f(p,t)dt \right| < \varepsilon, \forall p \in D$.

Proof: Exercise 6.74

Theorem 6.21: Suppose

(i) $f = f(p,t)$ is continuous on $\{(p,t) : p \in D, t \in [0,\infty)\}$,

(ii) $\int_0^\infty f(p,t)dt = F(p)$ is uniformly convergent on D.

Then F is continuous on D.

Proof: Exercise 6.75

Theorem 6.22: Suppose

(i) D has Jordan content ,

(ii) $f = f(p,t)$ is continuous on $\{(p,t) : p \in D , t \in [0,\infty)\}$,

(iii) $\int_0^\infty f(p,t)dt$ is uniformly convergent on D.

Then $\int_D \{\int_0^\infty f(p,t)dt\}dp = \int_0^\infty \{\int_D f(p,t)dp\}dt$.

Proof: Exercise 6.76

Theorem 6.23: Suppose

(i) $f(x,t), \dfrac{\partial f}{\partial x}(x,t)$ are continuous on $\{(x,t) : x \in [a,b], t \in [0,\infty)\}$,

(ii) $\int_0^\infty f(x_o,t)dt$ is convergent for some $x_o \in [a,b]$,

(iii) $\int_0^\infty \dfrac{\partial f}{\partial x}(x,t)dt$ is uniformly convergent on $[a,b]$.

Then $\dfrac{d}{dx}\int_0^\infty f(x,t)dt = \int_0^\infty \dfrac{\partial f}{\partial x}(x,t)dt$, $\forall\ x \in [a,b]$

Proof: Exercise 6.77

Note: The preceding theorems are still valid if the continuity in t is relaxed to integrability.

Theorem 6.24 (Weierstrass Comparison Test): Suppose

(i) $\int_a^b f(p,t)dt \rightrightarrows | \quad \forall \ [a,b] \subset [0,\infty)$,

(ii) $|f(p,t)| \leq M(t)$, $\forall \ p \in D$ and $\int_0^\infty M \in C$.

Then $\int_0^\infty f(p,t)$ is absolutely and uniformly convergent on D.

Proof: $\left| \int_{T_1}^{T_2} f(p,t)dt \right| \leq \int_{T_1}^{T_2} |f(p,t)dt \leq \int_{T_1}^{T_2} M$, if $T_2 \geq T$. The result follows from the Cauchy Criterion. □

Examples:

(1) $\int_1^\infty \frac{\cos(xt)}{t^2} dt$ is absolutely and uniformly convergent $-\infty < x < \infty$

since $\left| \frac{\cos(xt)}{t^2} \right| \leq \frac{1}{t^2}$ and $\int_1^\infty \frac{1}{t^2} dt = 1$.

(2) $\int_0^\infty e^{-xt} \cos t \ dt$ is absolutely convergent $0 < x < \infty$ and uniformly convergent $\delta < x < \infty$, if $\delta > 0$.

$x > 0 : |e^{-xt} \cos t| \leq e^{-xt}$ and $\int_0^\infty e^{-xt} dt = \frac{1}{x}$.

$x \geq \delta > 0 : |e^{-xt} \cos t| \leq e^{-xt} \leq e^{-\delta t}$ and $\int_0^\infty e^{-\delta t} dt = \frac{1}{\delta}$.

To evaluate the integral consider

$$\int_0^T e^{-xt} \cos t \ dt = e^{-xt} \sin t \Big|_0^T + x \int_0^T e^{-xt} \sin t \ dt$$

$$= (e^{-xt} \sin t - x e^{-xt} \cos t) \Big|_0^T - x^2 \int_0^T e^{-xt} \cos t \ dt$$

$$\therefore \quad (1 + x^2) \int_0^T e^{-xt} \cos t \, dt = e^{-xT}(\sin T - x \cos T) + x$$

$$\therefore \quad \int_0^\infty e^{-xt} \cos t \, dt = \frac{x}{1 + x^2}$$

Exercises:

6.78: Use Example 2 and some of the preceding theorems to show

(a) $\displaystyle \int_0^\infty t \, e^{-xt} \cos t \, dt = \frac{x^2 - 1}{(1 + x^2)^2}$, if $x > 0$

(b) $\displaystyle \int_0^\infty \{e^{-at} - e^{-bt}\} \frac{\cos t}{t} dt = \frac{1}{2} \log \left(\frac{1+b^2}{1+a^2} \right)$, if $0 < a \le b$.

6.79: Prove that each of the following integrals is uniformly convergent

for the range of values of x indicated:

(a) $\displaystyle \int_1^\infty \frac{dt}{t^2 + x^2}$, (all x) , (b) $\displaystyle \int_0^\infty \frac{dt}{t^2 + x^2}$, $(|x| \ge \delta > 0)$,

(c) $\displaystyle \int_0^\infty \frac{dt}{t^2 + x}$, $(x \ge \delta > 0)$, (d) $\displaystyle \int_0^\infty e^{-t} \cos(xt) dt$, (all x) ,

(e) $\displaystyle \int_0^\infty t^{10} e^{-t^2} \sin(xt) dt$, (all x) .

Tests for nonabsolute uniform convergence:

Theorem 6.25 (Dirichlet's Test): Suppose

(i) $\displaystyle \left| \int_0^T f(p,t) dt \right| \le B$, $0 \le T < \infty$, $\forall \, p \in D$,

(ii) $\displaystyle \lim_{t \to \infty} g(p,t) = 0$ uniformly on D and

$\displaystyle \int_0^\infty \left| \frac{\partial g}{\partial t}(p,t) \right| dt$ is uniformly convergent on D .

Then $\int_0^\infty f(p,t)g(p,t)dt$ is uniformly convergent on D.

[(ii) holds in particular if $g(p,t)$ is increasing or decreasing in t and $\lim_{t\to\infty} g(p,t) = 0$ uniformly on D.]

Proof: Let $F(p,T) = \int_0^T f(p,t)dt$; (i) $\Rightarrow$

$$|F(p,T)| \le B , \forall \cdot p \in D , 0 \le T < \infty$$

$$\left|\int_{T_1}^{T_2} f(p,t)g(p,t)dt\right|$$

$$= \left|F(p,T_2)g(p,T_2) - F(p,T_1)g(p,T_1) - \int_{T_1}^{T_2} F(p,T) \frac{\partial g}{\partial t}(p,t)dt\right|$$

$$\le B\{|g(p,T_2)| + |g(p,T_1)| + \int_{T_1}^{T_2} |\frac{\partial g}{\partial t}(p,t)|dt\}, \quad T_2 > T_1$$

Now use (ii) and the Cauchy Criterion.   □

Example 3: $\int_1^\infty e^{-xt} \frac{\sin t}{t} dt$ is uniformly convergent, $0 \le x < \infty$.

(i) $\left|\int_1^T \sin t \, dt\right| \le 2 , \forall x$ (x absent)

(ii) $\frac{e^{-xt}}{t}$ decreases to 0 as $t \to \infty \; \forall x \ge 0$

$\lim_{t\to\infty} \frac{e^{-xt}}{t} = 0$ uniformly on $[0,\infty)$ since $\left|\frac{e^{-xt}}{t}\right| \le \frac{1}{t}$.

Theorem 6.26 (Abel's Test): Suppose

    (i) $\int_0^\infty f(p,t)dt$ is uniformly convergent, $p \in D$,

    (ii) $|g(p,t)| \leq B$, $\int_0^\infty \left|\frac{\partial g}{\partial t}(p,t)\right|dt \leq B$ $\forall\ p \in D$.

          Then $\int_0^\infty f(p,t)g(p,t)dt$ is uniformly convergent on D.

[(ii) holds in particular if g(p,t) is either increasing or decreasing in t and $|g(p,t)| \leq B$ $\forall\ p \in D$.

Please note $\lim_{t\to\infty} g(p,t)$ exists $\forall\ p \in D$ but need not be uniformly convergent; the bound B must be uniform on D.]

Proof: Exercise 6.80

Example 4: $\int_0^\infty e^{-xt}\frac{\sin t}{t}dt$ is uniformly convergent, $0 \leq x < \infty$

    (i) $\int_0^\infty \frac{\sin t}{t}dt$ is uniformly convergent $0 \leq x < \infty$ since x is

          absent.

    (ii) e^{-xt} is decreasing in t, $|e^{-xt}| \leq 1$, $\forall\ x \geq 0$.

[Compare with Example 3. Can you use Dirichlet's Test for this Example.]

Application:

    As indicated in Exercise 6.78 improper integrals may often be evaluated by the introduction of a parameter and differentiating or integrating with respect to that parameter to obtain something simpler.

Examples:

(5)
$$\int_0^\infty \frac{\sin t}{t} = \frac{\pi}{2}$$

Proof: We have seen $F(x) = \int_0^\infty e^{-xt} \frac{\sin t}{t} dt$ is uniformly convergent,

$0 \le x < \infty$. $\therefore$ $F \in C[0,\infty)$, by Theorem 6.21. Also

$$\frac{d}{dx} \int_0^\infty e^{-xt} \frac{\sin t}{t} dt = - \int_0^\infty e^{-xt} \sin t \, dt \, , \quad x > 0$$

the last integral being uniformly convergent $x \ge \delta > 0$, by the Weierstrass

Test, since $|e^{-xt} \sin t| \le e^{-\delta t}$, $x \in [\delta,\infty)$.

Thus $F'(x) = - \int_0^\infty e^{-xt} \sin t \, dt = \frac{-1}{1 + x^2}$, $x > 0$, from integration by

parts (twice), and so

$$F(x) = - \tan^{-1} x + C \, , \quad x > 0$$

$$F \in C[0,\infty) \Rightarrow F(0) = F(0+) = C$$

Also $\lim_{x \to \infty} F(x) = 0$ (Why ?) $= - \frac{\pi}{2} + C$ $\therefore$ $C = \frac{\pi}{2}$

$$\therefore \quad F(0) = \int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$$

(6)
$$\int_0^\infty \frac{e^{-bx} - e^{-ax}}{x} dx = \log \left(\frac{b}{a}\right) \, , \, b \ge a > 0 \, .$$

Proof:
$$\int_0^\infty \frac{e^{-bx} - e^{-ax}}{x} dx = \int_0^\infty \int_a^b e^{-ux} \, du \, dx$$

$$= \int_a^b \int_0^\infty e^{-ux} \, dx \, du \quad \text{(by Theorem 6.22)}$$

$$= \int_a^b \frac{1}{u} \, du = \log \left(\frac{b}{a}\right)$$

Theorem 6.22 is applicable since by the Weierstrass Test $|e^{-ux}| \le e^{-ax}$,

$a \le u \le b$, so $\int_0^\infty e^{-ux} \, dx$ is uniformly convergent for u $\in [a,b]$ if $a > 0$.

(7) (i) $\displaystyle\int_0^\infty \frac{1}{(x^2 + a^2)^2}\, dx = \frac{\pi}{4a^3}$, $a \neq 0$.

(ii) $\displaystyle\int_0^\infty \frac{1}{x} \{\tan^{-1}(\frac{b}{x}) - \tan^{-1}(\frac{a}{x})\} dx = \frac{\pi}{2} \log(\frac{b}{a})$, $0 < a \leq b$.

Proof: $\displaystyle\int_0^\infty \frac{1}{x^2 + u^2}\, dx = \frac{1}{u} \tan^{-1} \frac{x}{u}\Big|_0^\infty = \frac{\pi}{2u}$, the integral being uniformly

convergent with respect to u, $|u| \geq \delta > 0$, by the Weierstrass Test since

$\dfrac{1}{x^2 + u^2} \leq \dfrac{1}{x^2 + \delta^2}$ if $|u| \geq \delta > 0$.

  (i) <u>Differentiate</u> with respect to u

$$- 2u \int_0^\infty \frac{1}{(x^2 + u^2)^2}\, dx = - \frac{\pi}{2u^2} ,\ u \neq 0$$

by Theorem 6.23 since the differentiated integral is also uniformly

convergent, $|u| \geq \delta > 0$, again by the Weierstrass Test. Thus

$$\int_0^\infty \frac{1}{(x^2 + a^2)^2}\, da = \frac{\pi}{4a^3} ,\quad a \neq 0 .$$

  (ii) <u>Integrate</u> with respect to u $(0 \notin [a,b])$

$$\frac{\pi}{2} \log(\frac{b}{a}) = \frac{\pi}{2} \int_a^b \frac{1}{u}\, du = \int_a^b \int_0^\infty \frac{1}{x^2 + u^2}\, dx\, du$$

$$= \int_0^\infty \int_a^b \frac{1}{x^2 + u^2}\, du\, dx \qquad \text{(Theorem 6.22)}$$

$$= \int_0^\infty \frac{1}{x} \{\tan^{-1}(\frac{b}{x}) - \tan^{-1}(\frac{a}{x})\} dx$$

(8) $\displaystyle\int_{0+}^{1} \frac{x^\alpha - 1}{\log x}\, dx = \log(\alpha+1)\ ,\quad \alpha > -1 \ .$

Proof:

$$F(\alpha) = \int_{0+}^{1} \frac{x^\alpha - 1}{\log x}\, dx \in C\ ,\ \alpha > -1$$

$$\therefore\ F'(\alpha) = \int_{0+}^{1} x^\alpha\, dx\ ,\ \alpha > -1\ ,\ \text{by Theorem 6.23 since this}$$

integral is uniformly convergent for $\alpha \geq -1 + \delta > -1$ by the

Weierstrass Test

$$|x^\alpha| \leq x^{-1+\delta},\ \int_{0+}^{1} x^{-1+\delta}dx = \frac{1}{\delta}$$

$$\therefore\ F'(\alpha) = \frac{1}{1+\alpha}\ \text{and}\ F(\alpha) = \log(\alpha+1) + C$$

$$\text{But}\quad 0 = F(0) = C,\ \text{so}\ F(\alpha) = \log(\alpha+1)$$

<u>Exercises</u>: [#6.81(e), 6.93(a), 6.94(a) *are important for Fourier series*]

6.81: Use the result $\displaystyle\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$ and elementary manipulations with

trigonometric formulae to show:

(a) $\displaystyle\frac{2}{\pi}\int_{0}^{\infty} \frac{\sin(ax)}{x}\, dx = \begin{cases} 1\ ,\ a > 0 \\ 0\ ,\ a = 0 \\ -1\ ,\ a < 0 \end{cases}$

(b) $\displaystyle\frac{2}{\pi}\int_{0}^{\infty} \frac{\sin x \cos(ax)}{x}\, dx = \begin{cases} 1\ ,\ |a| < 1 \\ \frac{1}{2}\ ,\ |a| = 1 \\ 0\ ,\ |a| > 1 \end{cases}$

(c) $\displaystyle\frac{2}{\pi}\int_{0}^{\infty} \frac{\cos x \sin(ax)}{x}\, dx = \begin{cases} -1\ ,\ a < -1 \\ -\frac{1}{2}\ ,\ a = -1 \\ 0\ ,\ -1 < a < 1 \\ \frac{1}{2}\ ,\ a = 1 \\ 1\ ,\ a > 1 \end{cases}$

(d) $\dfrac{2}{\pi} \displaystyle\int_0^\infty \left(\dfrac{\sin x}{x}\right)^2 dx = 1$

(e) $\displaystyle\lim_{n\to\infty} \int_0^\delta \dfrac{\sin(nx)}{x}\, dx = \dfrac{\pi}{2}$, $\displaystyle\lim_{n\to\infty} \int_\delta^\infty \dfrac{\sin(nx)}{x}\, dx = 0$, if $\delta > 0$.

6.82:   Show that $\displaystyle\int_0^\infty t^p\, e^{-t}\, dt$ is uniformly convergent for $p \in [0,M]$ if

$M > 0$ but not for $p \in [0,\infty)$.

6.83:   Prove $\displaystyle\int_{0+}^\infty \log\!\left(1 + \dfrac{a^2}{x^2}\right) dx = \pi a$, $a \geq 0$.

6.84:(a) $\displaystyle\int_0^\infty \dfrac{e^{-ax} - e^{-bx}}{x}\, \cos x \, dx = \dfrac{1}{2} \log\!\left(\dfrac{1+b^2}{1+a^2}\right)$, $a,\, b > 0$,

(b) $\displaystyle\int_0^\infty \dfrac{e^{-ax} - e^{-bx}}{x}\, \sin x \, dx = \tan^{-1} b - \tan^{-1} a$, $a,\, b > 0$

(c) Deduce from (b) $\displaystyle\int_0^\infty e^{-ax}\, \dfrac{\sin x}{x}\, dx = \dfrac{\pi}{2} - \tan^{-1} a$, $a > 0$

(d) Deduce from (c) $\displaystyle\int_0^\infty \dfrac{\sin x}{x}\, dx = \dfrac{\pi}{2}$

6.85:   Prove $\displaystyle\int_0^\infty \dfrac{(\tan^{-1} x)^2}{x^2}\, dx = \pi \log 2$

[Consider $F(a,b) = \displaystyle\int_0^\infty \dfrac{\tan^{-1}(ax)\, \tan^{-1}(bx)}{x^2}\, dx$

You will need to show $\displaystyle\int_0^\infty \dfrac{dx}{(1+a^2 x^2)(1+b^2 x^2)} = \dfrac{\pi}{2(a+b)}$

6.86:   Prove $\displaystyle\int_{0+}^\infty \left\{\log\!\left(1 + \dfrac{1}{x^2}\right)\right\}^2 dx = 4\pi \log 2$

6.87: (Frullani Integrals) Let $\phi \in C^1[0,\infty)$,

$$\lim_{u\to\infty} \phi(u) = A , \quad \lim_{u\to 0+} \phi(u) = B$$

Prove $\displaystyle\int_0^\infty \frac{\phi(bx) - \phi(ax)}{x}\,dx = (A - B)\log\left(\frac{b}{a}\right)$, $b \ge a > 0$.

[Show $\displaystyle\int_0^\infty \phi'(ux)dx$ is uniformly convergent, $u \in [a,b]$.]

Deduce (a) $\displaystyle\int_0^\infty \frac{e^{-bx} - e^{-ax}}{x}\,dx = \log\left(\frac{a}{b}\right)$, $b \ge a > 0$,

(b) $\displaystyle\int_{0+}^\infty \frac{1}{x^2}\left[e^{-ax}\{1 + x(a+k)\} - e^{-bx}\{1 + x(b+k)\}\right]dx$

$$= (b-a) + k\log\left(\frac{b}{a}\right) \text{ , } a, b > 0 .$$

6.88: $\displaystyle\int_0^\infty \frac{\sin(ax)\,\cos x}{x}\,dx = \begin{cases} 0 , & 0 \le a < 1 . \\ \dfrac{\pi}{4} & a = 1 \\ \dfrac{\pi}{2} , & a > 1 \end{cases}$

6.89: Prove $\displaystyle\int_0^1 x^x\,dx = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k^k}$

Approximate $\displaystyle\int_0^1 x^x\,dx$ so that the error does not exceed $.001$.

6.90: The error function (in statistics) is defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

(i) Show $\text{erf}(\infty) = 1$ (Exercise 6.43)

(ii) Express the following in terms of the error function:

(a) $\displaystyle\int_0^L e^{-1/s^2}\,ds$

(b) $\displaystyle\int_0^{'} x^2 e^{-x^2}\,dx$

(c) $\displaystyle\int_0^\infty e^{-xt^2}\,dt$

(d) $\displaystyle\frac{2}{\sqrt{\pi}}\int_0^{\text{ghanistan}} e^{-t^2}\,dt$

(iii) $\displaystyle\int_0^\infty \frac{1-e^{-ax^2}}{x^2}\,dx = \sqrt{\pi a}. \quad (a \ge 0)$

6.91:   Suppose

    (i) $f_n \in C[0,\infty)$, $n = 1, 2, \ldots,$

    (ii) $f_n(x) \to g(x)$ uniformly $x \in [0,R]$, $\forall R > 0$,

    (iii) $|f_n(x)| \leq M(x)$, $\int_0^\infty M \in C$,

    (iv) $\lim\limits_{n \to \infty} x_n = \infty.$

    Show $\lim\limits_{n \to \infty} \int_0^{x_n} f_n = \int_0^\infty g$

6.92:   Prove $\lim\limits_{n \to \infty} \int_0^n (1 - \frac{x}{n})^n x^k \, dx = k!$, $k = 0, 1, 2 \ldots$

    Guess the limits $\lim\limits_{n \to \infty} \int_0^n (1 - \frac{x}{n})^n e^{x/2} \, dx$,

    $\lim\limits_{n \to \infty} \int_0^n (1 + \frac{x}{n})^n e^{-2x} \, dx$ and prove your guesses are correct.

6.93:(a)   Prove that if $f \in C[0,1]$ there is a <u>step function</u> g on $[0,1]$ such that $|f(x) - g(x)| < \epsilon$, $\forall x \in [0,1]$ if $\epsilon > 0$. [g is a step function on $[0,1]$ if the interval can be partitioned into subintervals on each of which g is constant.]

   (b)   Prove that if $f \in C[0,1]$ there is a <u>polygonal function</u> g on $[0,1]$ such that $|f(x) - g(x)| < \epsilon$, $\forall x \in [0,1]$ if $\epsilon > 0$. [g is a polygonal function on $[0,1]$ if $g \in C[0,1]$ and $[0,1]$ can be partitioned into subintervals on each of which the graph of g is a straight line.]

6.94:(a)   If $\int_a^b f$ exists then there is a step function g on $[a,b]$ such that $\int_a^b |f - g| < \epsilon$, if $\epsilon > 0$.

(b) If $\int_0^\infty f$ is convergent then there is a step function g on $[0,\infty)$

with $g(x) = 0$ if $x > R$ for some $R > 0$ and $\int_0^\infty |f-g| < \epsilon$, if $\epsilon > 0$.

6.95: (Riemann-Lebesgue Lemma)

(a) If f is integrable on $[a,b]$ then

$$\lim_{n\to\infty} \int_a^b f(x) \sin(nx)dx = 0$$

[Prove it first if f is constant, then if f is a step function and then use Exercise 6.94(a) to prove it if $\int_a^b f$ exists.]

(b) If $\int_0^\infty f \in Abs\ C$ then $\lim_{n\to\infty} \int_0^\infty f(x) \sin(nx)dx = 0$.

6.96: Let $f(x) = e^{x^2} \int_x^\infty e^{-t^2} dt$. Show that $\int_0^\infty f$ is divergent.

6.97:(a) If $I_n = \int_0^\pi \cos^{2n}\theta\ d\theta$, $n = 0, 1, 2, \ldots$

Prove $I_n = \frac{2n-1}{2n} I_{n-1}$, $n = 1, 2, 3, \ldots$

(b) Prove

$$\frac{1}{\pi} \int_0^\pi \cos(x \cos \theta)d\theta = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots, \quad \text{all } x$$

(c) Approximate $\frac{1}{\pi} \int_0^\pi \cos(\cos \theta)d\theta$ so that the error is less than .0005.

Justify carefully any assertions you make.

REMARKS ON FOURIER SERIES

We shall need the following formulas:

$$\int_{-\pi}^{\pi} \sin(mx) \, \sin(nx) \, dx = \begin{cases} 0 & , \; m \neq n \\ \pi & , \; m, \, n = 1, \, 2, \, \dots \\ \pi & , \; m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx) \, \cos(nx) \, dx = \begin{cases} 0 & , \; m \neq n \\ \pi & , \; m = n \\ 2\pi & , \; m = n = 0 \end{cases} , \; m, \, n = 1, \, 2, \, \dots$$

$$\int_{-\pi}^{\pi} \sin(mx) \, \cos(nx) \, dx = 0, \; m, \, n = 0, \, 1, \, 2, \, \dots .$$

Now suppose $f(x) = \dfrac{1}{2} a_o + \sum_{k=1}^{\infty} \{a_k \cos kx + b_k \sin kx\}$, $-\pi \leq x \leq \pi$.

$$\int_{-\pi}^{\pi} f(x) \, \cos nx \, dx = \int_{-\pi}^{\pi} [\tfrac{1}{2} a_o + \sum_{k=1}^{\infty} \{a_k \cos kx + b_k \sin kx\}] \cos nx \, dx$$

$$\overset{*}{=} \frac{a_o}{2} \int_{-\pi}^{\pi} \cos nx \, dx + \sum_{k=1}^{\infty} a_k \int_{-\pi}^{\pi} \cos kx \cos nx \, dx + \sum_{k=1}^{\infty} b_k \int_{-\pi}^{\pi} \sin kx \cos nx \, dx$$

$$= \pi \, a_n$$

(A) $\quad \therefore \; a_n = \dfrac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx , \quad n = 0, \, 1, \, 2, \, \dots .$

Similarly

(B) $\quad b_n = \dfrac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx , \quad n = 1, \, 2, \, \dots .$

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* We have assumed here that convergence is such that the interchange of the integration and summation is valid.

<u>Definition</u>: For any function f on $[-\pi, \pi]$ such that the integrals in (A),

(B) exist

$$\frac{a_o}{2} + \sum_{k=1}^{\infty} \{a_k \cos kx + b_k \sin kx\}$$

is called the <u>Fourier</u> <u>Series</u> of f. We write this

$$f(x) \sim \frac{a_o}{2} + \sum_{k=1}^{\infty} \{a_k \cos kx + b_k \sin kx\}$$

The Fourier Series may or may not converge to $f(x)$.

A function f is <u>periodic</u> of period 2π if $f(x + 2\pi) = f(x)$, $\forall x \in R$. If f is periodic of period 2π and $\int_{-\pi}^{\pi} f$ exists then $\int_{-\pi+\alpha}^{\pi+\alpha} f$

exists and $\int_{-\pi}^{\pi} f = \int_{-\pi+\alpha}^{\pi+\alpha} f$, $\forall \alpha \in R$.

We denote

$$f(x_o+) = \lim_{x \to x_o +} f(x) \; , \; f(x_o-) = \lim_{x \to x_o -} f(x)$$

whenever the limits exist; further

$$f'_R(x_o) = \lim_{x \to x_o +} \frac{f(x) - f(x_o+)}{x - x_o} \; , \; f'_L(x_o) = \lim_{x \to x_o -} \frac{f(x) - f(x_o-)}{x - x_o} \; ,$$

the right and left derivatives of f at x_o.

<u>Theorem 6.27</u>: Suppose f is periodic of period 2π, $\int_{-\pi}^{\pi} f$ exists and

$$f(x) \sim \frac{a_o}{2} + \sum_{k=1}^{\infty} \{a_k \cos kx + b_k \sin kx\} \; .$$

Then

$$\frac{1}{2}\{f(x+) + f(x-)\} = \frac{a_o}{2} + \sum_{k=1}^{\infty} \{a_k \cos kx + b_k \sin kx\}$$

at any point x where $f'_R(x)$ and $f'_L(x)$ exist.

Proof: $S_n(x) \overset{\text{def}}{=} \dfrac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos kx + b_k \sin kx]$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi} \sum_{k=1}^{n} \int_{-\pi}^{\pi} f(t) [\cos kt \cos kx + \sin kt \sin kx]dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) [\frac{1}{2} + \sum_{k=1}^{n} \cos k(t-x)]dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin(n+\frac{1}{2})(t-x)}{2 \sin \frac{1}{2}(t-x)} dt \qquad [\text{cf. Example 5, p. 367}]$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) \frac{\sin(n+\frac{1}{2})t}{2 \sin \frac{1}{2} t} dt$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \{f(x+t) + f(x-t)\} \frac{\sin(n+\frac{1}{2})t}{2 \sin \frac{1}{2} t} dt$$

In the case of the function $f(x) \equiv 1$, $a_0 = 2$ and $a_k = b_k = 0$, $k = 1,2,\cdots$, so that $S_n(x) = 1$ for all n and therefore from the preceding calculation

$$1 = \frac{1}{\pi} \int_{0}^{\pi} \frac{\sin(n + \frac{1}{2})t}{\sin \frac{1}{2} t} dt, \quad n = 0, 1, 2, \cdots .$$

Thus, for any function $f(x)$,

$$S_n(x) - \frac{1}{2} \{f(x+) + f(x-)\} = \frac{1}{\pi} \int_{0}^{\pi} F(t) \sin(n + \frac{1}{2})t \, dt$$

where $F(t) = [f(x+t) + f(x-t) - f(x+) - f(x-)] / 2 \sin \frac{1}{2} t$.

Note that $F(0+)$ exists if $f_R'(x)$ and $f_L'(x)$ exist so that $\int_0^\pi F$ exists and

hence $\lim_{n \to \infty} \int_{0}^{\pi} F(t) \sin(n + \frac{1}{2})t \, dt = 0$ by the Riemann − Lebesgue Lemma
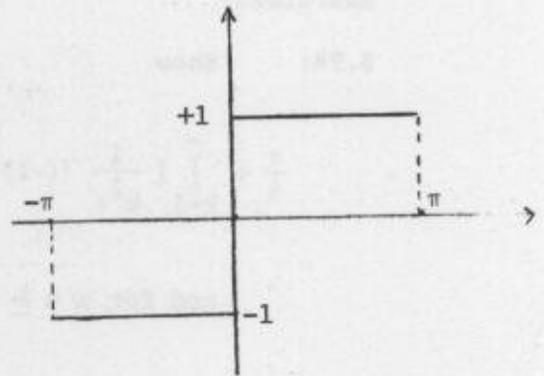
(Exercise 6.95 (a)); Therefore $\lim_{n \to \infty} S_n(x) = \frac{1}{2} \{f(x+) + f(x-)\}$.

Note: If f has a left and right derivative at each point in its domain the Fourier Series converges to $f(x)$ at a point x where f is continuous and to the point midway between $f(x+)$ and $f(x-)$ if f is discontinuous at x.

A more detailed discussion of Fourier series is given in Mathematics 437 and 443.

Example 1:

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\[2mm] 1, & 0 < x < \pi \end{cases}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \sin nx \, dx = \frac{2}{n\pi} [1 - (-1)^n]$$

$$= \begin{cases} 0 & , \text{ n even} \\[3mm] \dfrac{4}{n\pi} & , \text{ n odd} \end{cases}$$

$$\therefore \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin(2k-1)x = \begin{cases} 1, & 0 < x < \pi \\ 0, & x = 0 \\ -1, & -\pi < x < 0 \end{cases}$$
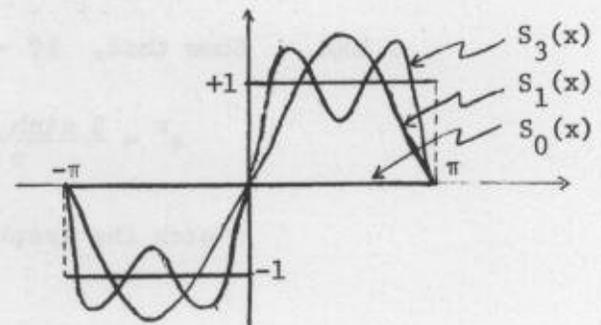
By periodicity the series converges to -1 on $((2m-1)\pi, 2m\pi)$, to 1 on $(2m\pi, (2m+1)\pi)$ and to 0 at $m\pi$.

$$S_o(x) = 0$$

$$S_1(x) = \frac{4}{\pi} \sin x$$

$$S_2(x) = \frac{4}{\pi} \sin x$$

$$S_3(x) = \frac{4}{\pi} (\sin x + \frac{1}{3} \sin 3x)$$

$$x = \frac{\pi}{2} : 1 = \frac{4}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k-1}$$
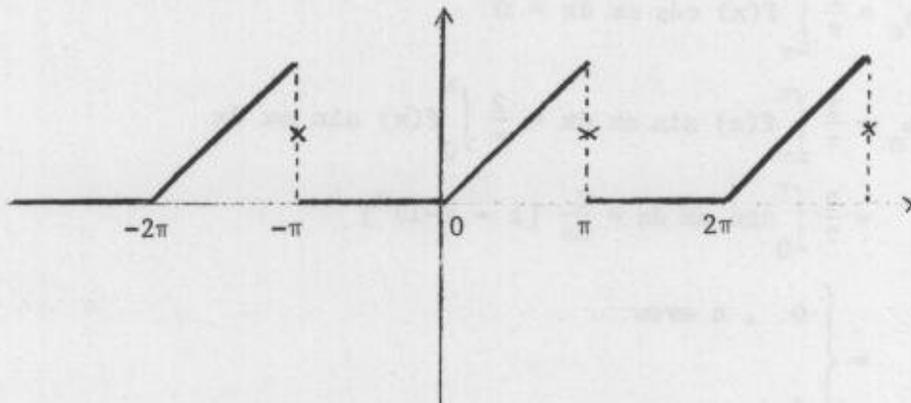
i.e. $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots$ as we have seen before.

Exercises:

6.98:   Show

$$\frac{\pi}{4} + \sum_{k=1}^{\infty} [\frac{1}{k^2 \pi} \{(-1)^k - 1\} \cos kx + \frac{(-1)^{k+1}}{k} \sin kx] = \begin{cases} 0, & -\pi < x \le 0 \\ \\ x, & 0 \le x < \pi \end{cases}$$

and for $x = \pm \pi$ the sum of the series is $\frac{\pi}{2}$.

6.99:   Show that for all x

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos 2kx}{4k^2 - 1}$$

In particular $\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} = \frac{1}{2}$, $\sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2 - 1} = \frac{1}{2} - \frac{\pi}{4}$.

6.100:   Show that,   if $- \pi < x < \pi$,

$$e^x = \frac{2 \sinh \pi}{\pi} [\frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{1+k^2} (\cos kx - k \sin kx)]$$

Sketch the graph of the sum on $[-3\pi, 3\pi]$.

Deduce $\dfrac{\pi}{2} \coth \pi = \dfrac{1}{2} + \displaystyle\sum_{k=1}^{\infty} \dfrac{1}{1+k^2}$.
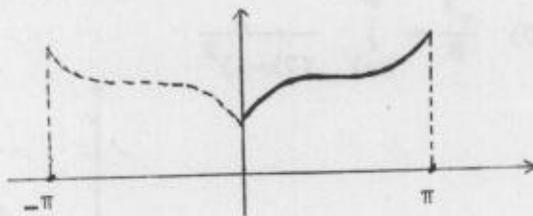
A function f is said to be <u>odd</u> if $f(-x) = -f(x) \;\forall\; x$ and <u>even</u> if $f(-x) = f(x), \;\forall\; x$.

$$f \text{ odd } \Rightarrow \int_{-\alpha}^{\alpha} f = 0$$

$$f \underline{\text{ odd }} \Rightarrow \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0 \;\Rightarrow\; a_n = 0$$
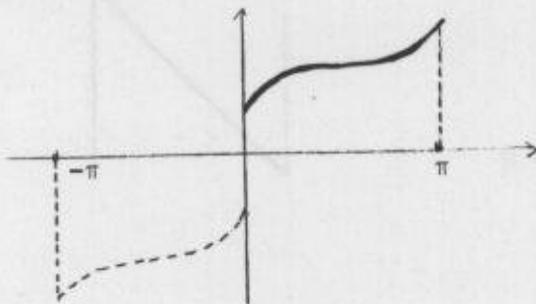
$$f \underline{\text{ even }} \Rightarrow \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0 \;\Rightarrow\; b_n = 0$$

Thus if we require a <u>cosine</u> expansion of $f(x)$, $0 < x < \pi$ define f as an <u>even</u> function on $[-\pi, \pi]$.

$$f(-x) = f(x)$$

If a <u>sine</u> expansion is required define f as an <u>odd</u> function on $[-\pi, \pi]$
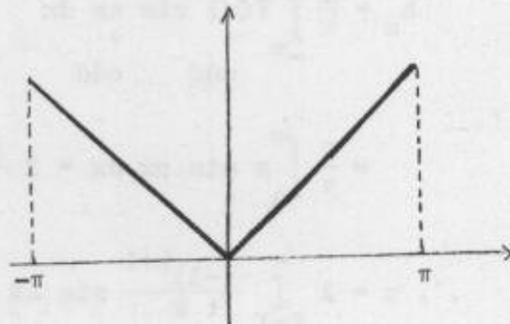

$$f(-x) = -f(x)$$

<u>Examples:</u>

(2) $\quad f(x) = \begin{cases} x, & 0 < x < \pi \\[2mm] -x, & -\pi < x < 0 \end{cases}$

i.e. $f(x) = |x| \quad , \; -\pi < x < \pi$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0$$

$$\underset{\text{even}}{\uparrow} \quad \underset{\text{odd}}{\uparrow}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$\underset{\text{even}}{\uparrow} \quad \underset{\text{even}}{\uparrow}$$

$$= \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx = \begin{cases} \pi, & n = 0 \\ \\ \frac{2}{\pi} \frac{(-1)^n - 1}{n^2}, & n = 1, 2, \ldots \end{cases}$$

$$\therefore |x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}$$

In particular (x=0) $\quad \dfrac{\pi^2}{8} = \displaystyle\sum_{k=1}^{\infty} \dfrac{1}{(2k-1)^2}$

(3)

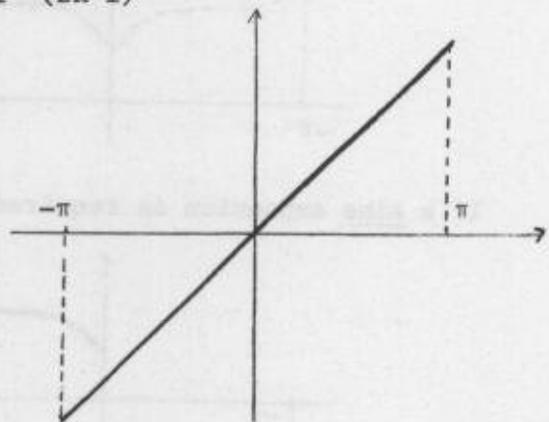$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \\ x, & -\pi < x < 0 \end{cases}$$

i.e. $f(x) = x, \quad -\pi < x < \pi$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0$$

$$\underset{\text{odd}}{\uparrow} \quad \underset{\text{even}}{\uparrow}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$\underset{\text{odd}}{\uparrow} \quad \underset{\text{odd}}{\uparrow}$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x \sin nx \, dx = 2 \frac{(-1)^{n+1}}{n}$$

$$\therefore x = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx, \quad -\pi < x < \pi$$

Exercises:

6.101: Show that both of the following series have sum e^x, $0 < x < \pi$ and sketch the graph of their sums on $[-3\pi, 3\pi]$.

(i) $\dfrac{1}{\pi} (e^{\pi} - 1) + \dfrac{2}{\pi} \displaystyle\sum_{k=1}^{\infty} \dfrac{(-1)^k e^{\pi} - 1}{1 + k^2} \cos kx$

(ii) $\dfrac{2}{\pi} \displaystyle\sum_{k=1}^{\infty} \dfrac{k}{1 + k^2} [1 - (-1)^k e^{\pi}] \sin kx$

6.102: Prove that

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos kx , \quad -\pi \le x \le \pi$$

and hence

$$\frac{\pi^2}{12} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} , \quad \frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Sketch the graph of the sum on $[-3\pi, 3\pi]$.

6.103: Show

(i) $\pi - x = \dfrac{\pi}{2} + \dfrac{4}{\pi} \displaystyle\sum_{k=1}^{\infty} \dfrac{\cos(2k-1)x}{(2k-1)^2} , \quad 0 < x < \pi$

(ii) $\pi - x = 2 \displaystyle\sum_{k=1}^{\infty} \dfrac{\sin kx}{k} , \quad 0 < x < \pi$

What is the sum of the series outside $(0, \pi)$?

6.104: Show

$$\int_{0+}^{1} \frac{1}{x} \log(1+x)\,dx = \frac{\pi^2}{12} ,$$

$$\int_{0+}^{1} \frac{1}{x} \log(1-x)\,dx = - \frac{\pi^2}{6} .$$

6.105:   Show

$$\cos x = \frac{8}{\pi} \sum_{k-1}^{\infty} \frac{k}{4k^2-1} \sin 2kx \ , \quad 0 < x < \pi$$

What is the sum for other values of x ?

References for Chapter VI

R.C. Buck: Advanced Calculus (Chapters 2,3,4).

R.G. Bartle: The Elements of Real Analysis (Chapters 3,6,7).

W. Rudin: Principles of Mathematical Analysis (Chapters 3,7).

K. Knopp: Theory and application of infinite series.

T.J. I'A. Bromwich: An introduction to the theory of infinite series.

THE PROFESSOR'S SONG

Words by Tom Lehrer - Tune: "If You Give Me Your Attention"
    from *Princess Ida* (Gilbert and Sullivan)

If you give me your attention, I will tell you what I am.
I'm a brilliant math'matician–also something of a ham.
I have tried for numerous degrees, in fact I've one of each;
Of course that makes me eminently qualified to teach.
I understand the subject matter thoroughly, it's true,
And I can't see why it isn't all as obvious to *you*.
Each lecture is a masterpiece, meticulously planned,
Yet everybody tells me that I'm hard to understand,
        And I can't think why.

My diagrams are models of true art, you must agree,
And my handwriting is famous for its legibility.
Take a word like "minimum" (to choose a random word),(*)
For anyone to say he cannot read that, is absurd.
The anecdotes I tell get more amusing every year,
Though frankly, what they go to prove is sometimes less than clear,
And all my explanations are quite lucid, I am sure,
Yet everybody tells me that my lectures are obscure,
        And I can't think why.

Consider for example, just the force of gravity:
It's inversely proportional to something–let me see–
It's r^3 - no, r^2 - no, it's just r, I'll bet–
The sign in front is plus–or is it minus, I forget,
Well, anyway, there *is* a force, of that there is no doubt.
All these formulas are trivial if you only think them out.
Yet students tell me, "I have memorized the whole year through
Ev'rything you've told us, but the problems I can't do."
        And I can't think why!

(*) This was performed at a blackboard, and the professor wrote:
        ∿∿∿∿∿∿∿∿∿∿∿∿∿∿∿∿