MATHEMATICS 118

LECTURE NOTES

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(Pages 153-340)

VII SOME SPECIAL FUNCTIONS

§7.1. Logarithm and Exponential.

We have seen that

$$D(x^{\alpha+1}/\alpha + 1) = x^{\alpha}$$

if $\alpha \in \mathbb{Q}, \alpha \neq -1$ for all x in the domain of $x^{\alpha+1}/\alpha + 1$ with the possible exception of $x = 0$. The Fundamental Theorem of Calculus allows us to say that, if $a > 0, x > 0$,
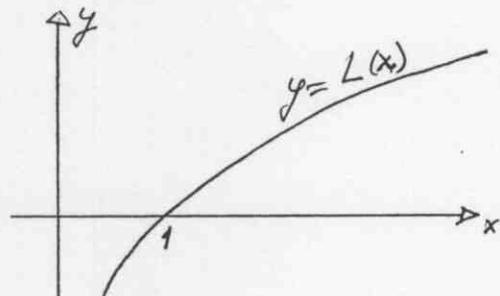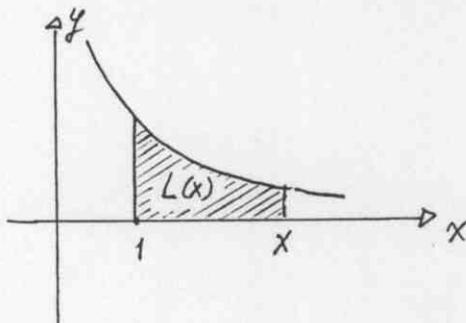
$$D\int_a^x t^{-1} dt = x^{-1}.$$

For definiteness, choose $a = 1$ and define

$$L(x) = \int_1^x \frac{1}{t} \, dt \quad x > 0$$

NOTES:

(i) $L'(x) = \frac{1}{x}$, for all $x > 0$

(ii) L is continuous and increasing on $(0, \infty)$

(iii) $L(1) = 0, L(x) < 0$, if $0 < x < 1$ and $L(x) > 0$ if $x > 1$.

The most important property of this function is the following:

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THEOREM 7.1.1. *If* a, b *are positive, then*

$$L(ab) = L(a) + L(b).$$

PROOF: Consider $f(x) = L(ax), \ x > 0$.

By the Chain Rule $f'(x) = \frac{1}{ax}a = \frac{1}{x} = L'(x)$. Therefore $f(x) = L(x) + c$ so that, setting $x = 1$,

$$L(a) = f(1) = L(1) + c = c, \quad \text{since} \quad L(1) = 0, \quad \text{giving}$$

$$f(x) = L(x) + L(a)$$

$$L(ax) = L(x) + L(a).$$

$\square$

COROLLARY 7.1.2. *If* $a > 0$ *and* $\alpha \in \mathbb{Q}$, *then*

$$L(a^{\alpha}) = \alpha L(a).$$

PROOF:

STEP 1. $L(a^n) = nL(a), \quad n = 1, 2, \ldots,$ by induction from Theorem 7.1.1. Also, from Theorem 7.1.1,

$$0 = L(1) = L(aa^{-1}) = L(a) + L(a^{-1}), \quad \text{so that}$$

$$L(a^{-1}) = -L(a).$$

Therefore $L(a^{-n}) = L((a^{-1})^n) \stackrel{\text{Step 1}}{=} nL(a^{-1}) = -nL(a)$ giving

STEP 2. $L(a^n) = nL(a), n = 0, \pm 1, \pm 2, \ldots$.

Now, if $b^n = a$

(i.e. $b = a^{1/n}$)   then

$$nL(b) = L(b^n) = L(a),$$

from Step 2 so that

STEP 3. $L(a^{1/n}) = \frac{1}{n} L(a)$ $n = 1, 2, 3, \ldots$.

Finally

$$L(a^{m/n}) = L((a^{1/n})^m) = mL(a^{1/n}), \quad \text{by Step 2}$$

$$= \frac{m}{n} L(a), \quad \text{by Step 3,} \quad \begin{matrix} m = 0, \pm 1, \pm 2 \\ n = 1, 2, \ldots \end{matrix} .$$

Therefore $L(a^\alpha) = \alpha L(a)$.

$\square$

COROLLARY 7.1.3. *The range of* L *is* $\mathbb{R} = (-\infty, \infty)$,

$$\lim_{x \to \infty} L(x) = \infty, \quad \lim_{x \to 0+} L(x) = -\infty.$$

PROOF:

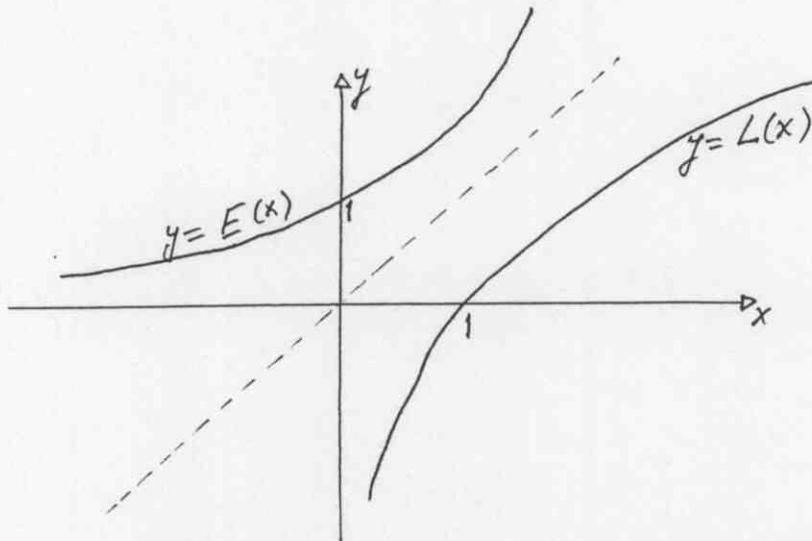$$x > 2^n \Rightarrow L(x) > L(2^n), \quad \text{since} \quad L \quad \text{is increasing}$$
$$= nL(2) \quad \text{and} \quad L(2) > 0$$
$$\Rightarrow \lim_{x \to \infty} L(x) = \infty.$$

Similarly

$$0 < x < 2^{-n} \Rightarrow L(x) < L(2^{-n}) = -nL(2)$$

$$\Rightarrow \lim_{x \to 0+} L(x) = -\infty.$$

The Intermediate Value Theorem for continuous functions implies that the range of L is all of $\mathbb{R}$.

$\square$

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Since L is increasing or $(0, \infty)$ with range $(-\infty, \infty)$, L has an inverse function E increasing on $(-\infty, \infty)$ with range $(0, \infty)$

$$E(L(x)) = x, \quad 0 < x < \infty.$$
$$L(E(x)) = x, \quad -\infty < x < \infty.$$

COROLLARY 7.1.4. If $a, b \in \mathbb{R}$, then

$$E(a)E(b) = E(a + b).$$

PROOF: From Theorem 7.1.1,

$$L(E(a)E(b)) = L(E(a)) + L(E(b)) = a + b = L(E(a + b))$$

and therefore $E(a)E(b) = E(a + b)$ since L is one-to-one.

$\square$

COROLLARY 7.1.5. If $a \in \mathbb{R}$, $\alpha \in \mathbb{Q}$, then

$$E(a)^\alpha = E(\alpha a).$$

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PROOF: From Corollary 7.1.2,

$$L\big(E(a)^\alpha\big) = \alpha L\big(E(a)\big) = \alpha a = L\big(E(\alpha a)\big).$$

□

If $e = E(1)$, then Corollary 7.1.5 shows

$$e^\alpha = E(\alpha), \quad \alpha \in \mathbb{Q}.$$

We thus define e^x for *all* $x \in \mathbb{R}$.

DEFINITION 7.1.6.

$$e^x = E(x), \quad x \in \mathbb{R}.$$

This motivates calling $L(x)$ the *natural logarithm* of x and writing

$$L(x) = \log x = \ell n\, x = \log_e x.$$

The number $e = E(1) \simeq 2.718$ is called the *natural base* for logarithms; the notations

$$E(x) = e^x = \exp(x)$$

are often used and exp is called the *exponential function*. More generally, Corollary 7.1.5 allows us to define b^x for any $b > 0$ and any $x \in \mathbb{R}$.

$$(e^a)^\alpha = e^{\alpha a} \quad \alpha \in \mathbb{Q}.$$

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Thus we define

$$(e^a)^x = e^{xa} \quad a, x \in \mathbb{R}$$

and with $b = e^a$ or, equivalently, $a = \log b$ we obtain the definition

DEFINITION 7.1.7.

$$b^x = e^{x \log b}, \quad b > 0 \quad x \in \mathbb{R}.$$

The inverse of the function thus defined is the *logarithm to base* b :

$$b^x = y \iff \log_b y = x \quad 0 < b \neq 1.$$

Clearly

$$y = b^x = e^{x \log b}$$

implies

$$x = log_b \, y \quad \text{and} \quad x \log \, b = log \, y$$

$$(\text{`log' is the natural logarithm}).$$

Thus

$$\log_b y = \frac{\log y}{\log b}, \quad 0 < b \neq 1.$$

PROPOSITION 7.1.8. $b > 0, p, q \in \mathbb{R}$

$$\implies \quad (i) \quad b^p b^q = b^{p+q}$$

$$(ii) \quad (b^p)^q = b^{pq}.$$

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PROOF: Exercise 7.2.2.

THEOREM 7.1.9. $De^x = e^x$.

PROOF: Since $D \log x = \frac{1}{x} > 0$, if $x > 0$, the exponential function, as the inverse of the natural logarithm, has from Proposition 5.5.6 a derivative at every point in its domain

$$y = e^x \implies \log y = x \implies \frac{1}{y} \frac{dy}{dx} = 1 \quad \text{(Chain Rule)}$$
$$\implies \frac{dy}{dx} = y.$$

$\square$

COROLLARY 7.1.10. $Db^x = b^x \log b, \quad b > 0$.

PROOF: Exercise 7.2.3.

The function $x^\alpha = e^{\alpha \log x}$ is now defined for all $\alpha \in \mathbb{R}$ and $x > 0$.

COROLLARY 7.1.11. $Dx^\alpha = \alpha x^{\alpha-1}$.

PROOF: Exercise 7.23.

PROPOSITION 7.1.12.

$$(1 + \frac{1}{n})^n \le e \le (1 + \frac{1}{n})^{n+1}.$$

PROOF: By definition e is the number such that $1 = \log e$,

$$1 = \int_1^e \frac{1}{t} \, dt.$$

Now $\log(1 + \frac{1}{n}) = \int_1^{1+\frac{1}{n}} \frac{1}{t}\,dt$

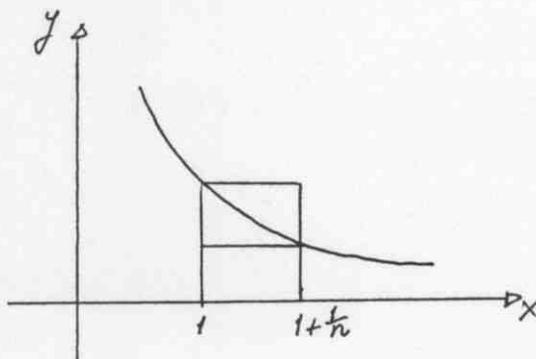and $\frac{1}{1+\frac{1}{n}} \leq \frac{1}{t} \leq 1$, if $1 \leq t \leq 1 + \frac{1}{n}$,

$$\implies \frac{1}{1+\frac{1}{n}} \cdot \frac{1}{n} \leq \log(1+\frac{1}{n}) \leq 1 \cdot \frac{1}{n}$$

$$\implies \frac{1}{n+1} \leq \log(1+\frac{1}{n}) \leq \frac{1}{n}$$

$$\implies 1 \leq (n+1)\log(1+\frac{1}{n}) \quad \text{and} \quad n\log(1+\frac{1}{n}) \leq 1$$

$$\implies 1 \leq \log[(1+\frac{1}{n})^{n+1}] \quad \text{and} \quad \log[(1+\frac{1}{n})^n] \leq 1$$

$$\implies e \leq (1+\frac{1}{n})^{n+1} \quad \text{and} \quad (1+\frac{1}{n})^n \leq e.$$

$\square$

COROLLARY 7.1.13.

$$\lim_{n\to\infty} (1+\frac{1}{n})^n = e.$$

PROOF: Proposition 7.1.12

$$\implies |(1+\frac{1}{n})^n - e| \leq (1+\frac{1}{n})^{n+1} - (1+\frac{1}{n})^n$$

$$= (1+\frac{1}{n})^n(1+\frac{1}{n} - 1) \leq e\,\frac{1}{n}.$$

$$\therefore \quad \lim_{n\to\infty} (1+\frac{1}{n})^n = e.$$

$\square$

NOTE: We saw in Examples 2.3.4 that $lim_{n\to\infty}(1 + \frac{1}{n})^n$ exists and defined e to be its limit. Corollary 7.1.13 shows that this is consistent with defining e by $1 = \log e$. In fact it is the case that for all $x \in \mathbb{R}$

$$e^x = lim_{n\to\infty}(1 + \frac{x}{n})^n \quad \text{(Exercise 7.12)}.$$

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THEOREM 7.1.14. $\lim_{x\to\infty} x^\alpha e^{-x} = 0,$ if $\alpha \in \mathbb{R}.$

COROLLARY 7.1.15. $\lim_{x\to\infty} x^\beta e^x = \infty,$ if $\beta \in \mathbb{R}.$

COROLLARY 7.1.16. $\lim_{x\to\infty} x^{-\gamma} \log x = 0,$ if $\gamma > 0.$

COROLLARY 7.1.17. $\lim_{x\to 0+} x^\delta \log x = 0,$ if $\delta > 0.$

PROOF: Consider $f(x) = x^{\alpha+1} e^{-x}$

$$f'(x) = (\alpha+1)x^\alpha e^{-x} - x^{\alpha+1} e^{-x}$$

$$= x^\alpha e^{-x}[(\alpha+1) - x] < 0, \quad \text{if} \quad x > \alpha + 1.$$

$\therefore f(x)$ is decreasing if $x > \alpha + 1$

$\therefore \quad 0 < f(x) < M = f(\alpha+1)$ if $x > \alpha + 1$ (case $\alpha + 1 \geq 0$)

$\quad 0 < f(x) < M = f(0)$ if $x > 0$ (case $\alpha + 1 < 0$)

$\therefore \quad 0 < x^{\alpha+1} e^{-x} < M,$ for x sufficiently large

$\therefore \quad 0 < x^\alpha e^{-x} < \dfrac{M}{x},$ for x sufficiently large

$\therefore \quad \lim_{x\to\infty} x^\alpha e^{-x} = 0.$

$\square$

Corollary 7.1.15 follows from $x^\beta e^x = \frac{1}{x^{-\beta} e^{-x}}.$

Corollary 7.1.16 is clear if we take $x = e^{t/\gamma}$ so that

$x^{-\gamma} \log x = e^{-t} t/\gamma \longrightarrow 0,$ as $t \longrightarrow \infty$ (i.e. $x \to \infty$).   Similarly

Corollary 7.1.17 follows from the substitution $x = e^{-t/\delta}.$ $\square$

SUMMARY:

(i) $\log x = \int_1^x \frac{1}{t} dt,$ $x > 0.$

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(ii) $D \log x = \frac{1}{x}, \quad x > 0.$

(iii) $\log(ab) = \log a + \log b, \quad a > 0, \quad b > 0.$

(iv) $\log(a^x) = x \log a$

(v) $e^{\log x} = x, \quad \log(e^x) = x$

(vi) $D e^x = e^x$

(vii) $b^x = e^{x \log b}$

(viii) $e^a e^b = e^{a+b}$

(ix) $(e^a)^b = e^{ab}.$
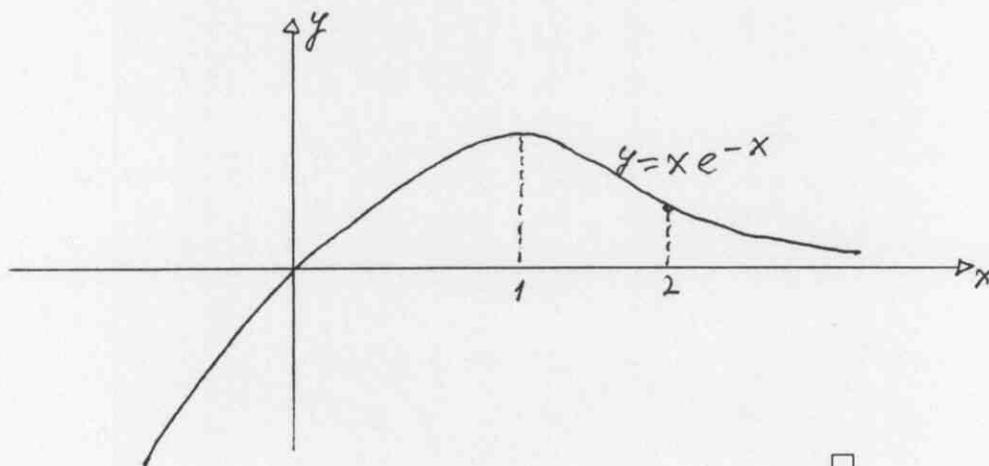
EXAMPLE 7.1.18: Sketch the graph of f where

$$f(x) = xe^{-x}, \quad -\infty < x < \infty.$$

$$f(x) \begin{cases} < 0, x < 0 & \lim_{x \to -\infty} f(x) = -\infty \\ > 0, x > 0 & \lim_{x \to \infty} f(x) = 0 \end{cases}$$

$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} \begin{cases} > 0, & \text{if} \quad x < 1 \quad \therefore f \quad \text{increasing on} \quad (-\infty, 1] \\ < 0, & \text{if} \quad x > 1 \quad \therefore f \quad \text{decreasing on} \quad [1, \infty) \end{cases}$$

$$f''(x) = -2e^{-x} + xe^{-x} = (x-2)e^{-x} \begin{cases} < 0, & \text{if} \quad x < 2 \quad \therefore \quad f \quad \text{concave down on} \quad (-\infty, 2] \\ > 0, & \text{if} \quad x > 2 \quad \therefore \quad f \quad \text{concave up on} \quad [2, \infty). \end{cases}$$

$\Box$

EXAMPLE 7.1.19: Let $f(x) = \log(x\sqrt{1-x})$. Find the domain of f and find $f'(x)$.

$f(x)$ is defined if $x\sqrt{1-x} > 0$ i.e. $x > 0$ and $x < 1$. Therefore the domain of f is $(0,1)$. To simplify the differentation, notice

$$f(x) = \log x + \frac{1}{2} \log(1-x), \quad 0 < x < 1$$
$$\therefore \quad f'(x) = \frac{1}{x} + \frac{1}{2} \frac{1}{1-x} (-1) = \frac{1}{x} - \frac{1}{2(1-x)}.$$

$\square$

The process of *logarithmic differentation* often simplifies the calculation of derivatives. It uses the formula

$$D \log |f(x)| = \frac{f'(x)}{f(x)}$$

when $f'(x)$ exists and $f(x) \neq 0$. When $f(x) > 0$, $\log|f(x)| = \log f(x)$ and, when $f(x) < 0$, $\log|f(x)| = \log(-f(x))$ so that the formula follows from the Chain Rule in both cases. If
$f(x) = (x-a_1)^{\alpha_1} \ldots (x-a_n)^{\alpha_n} / (x-b_1)^{\beta_1} \ldots (x-b_m)^{\beta_m}$ then

$$\log|f(x)| = \alpha_1 \log|x-a_1| + \cdots + \alpha_n \log|x-a_n| - \beta_1 \log|x-b_1| \cdots - \beta_m \log|x-b_m|$$

giving

$$\frac{f'(x)}{f(x)} = \frac{\alpha_1}{x-a_1} + \cdots + \frac{\alpha_n}{x-a_n} - \frac{\beta_1}{x-b_1} \cdots - \frac{\beta_m}{x-b_m}, \quad x \neq a_i, b_i.$$

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EXAMPLE 7.1.20:

$$f(x) = \frac{(x+2)^2(x-3)}{(x+1)^3(x-9)^4(x+10)}$$

$$\implies f'(x) = f(x)\left[\frac{2}{x+2} + \frac{1}{x-3} - \frac{3}{x+1} - \frac{4}{x-9} - \frac{1}{x+10}\right].$$

$\square$

EXAMPLE 7.1.21:

$$\int_0^{\log 5} e^{-x}dx = -e^{-x}\big|_0^{\log 5} = -e^{-\log 5} + e^0$$

$$= -e^{\log(\frac{1}{5})} + e^0 = -\frac{1}{5} + 1 = \frac{4}{5}.$$

$\square$

EXAMPLE 7.1.22:

$$D2^x = D(e^{\log 2})^x = De^{x\log 2} = (\log 2)e^{x\log 2} = (\log 2)2^x.$$

Alternatively,

$$y = 2^x \implies \log y = x\log 2$$
$$\implies \frac{1}{y}\frac{dy}{dx} = \log 2$$
$$\implies \frac{dy}{dx} = (\log 2)y$$

$\square$

EXAMPLE 7.1.23:

$$D\log_{10}|x| = D\left(\frac{\log|x|}{\log 10}\right) = \frac{1}{x\log 10}.$$

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Alternatively,

$$y = \log_{10} |x| \implies 10^y = x$$

$$\implies y \log 10 = \log |x|$$

$$\implies \frac{dy}{dx} \log 10 = \frac{1}{x}$$

$$\implies \frac{dy}{dx} = \frac{1}{x \log 10}.$$

$\square$

Problems

7.1. Find $f'(x)$:

    (a) $f(x) = 3^x$,   (b) $f(x) = x^x$,   (c) $f(x) = \log(x^2 + 3)$,

    (d) $f(x) = e^{\log x}$.

    [Answer: (b) $f'(x) = x^x(1 + \log x)$.]

7.2. Solve for x

    (a) $\log x = 0$,   (b) $\log x = 2$,

    (c) $(2 - \log x)\log x = 0$,

    (d) $\lim_{h \to 0} \frac{\log(x+h) - \log x}{h} = 3$,   (e) $e^{x^2+1} = 3$.

7.3. Sketch the curve $y = xe^{-x^2}$ with special attention to monotonicity, concavity and asymptotic behaviour,

7.4. Sketch the curve $y = x^x, x > 0$, paying careful attention to the behaviour of the curve near $x = 0$.

7.5. Find the derivatives of each of the following

    (a) e^{-2x},   (b) $3e^{x^2}$,   (c) $\frac{e^{1/x}}{x}$,

    (d) $\log(x^2 e^x)$,   (e) $\int_0^x e^{t^2} dt$.

7.6. Find the inverse functions:

    (a) $f(x) = \log(1 + x^3)$,   (b) $f(x) = 1 + e^x$.

7.7. Find

    (a) $\int_1^e \frac{dx}{x}$,   (b) $\int_1^e \frac{\log x}{x} dx$,   (c) $\int_e^{e^2} \frac{1}{x(\log x)^2} dx$,

    (d) $\int_0^{\log 2} e^{-x} dx$,   (e) $\int_{(\log 2)^2}^4 e^{\sqrt{x}}/\sqrt{x} \, dx$.

    Simplify your answers.

7.8. A particle moves along a straight line with acceleration at time t sec. given by $a(t) = -\frac{1}{(t+1)^2}$ cm/(sec)^2. Find the distance travelled by the particle between $t = 0$ and $t = 5$ given that the initial velocity is 3 cm/sec.

7.9. Find

(a) $\log_2 32$, (b) $\log_2(\frac{1}{32})$, (c) $\log_{\frac{1}{2}} 32$, (d) $\log_{32} 2$,

(e) $\log_{32} 64$.

7.10. (a) Show $\lim_{h \to 0} \frac{\log(1+h)}{h} = 1$.

(*Hint.* Consider $f'(1)$, if $f(x) = \log x$).

(b) Use (a) to show $\lim_{h \to 0}(1 + h)^{1/h} = e$.

(c) To the accuracy permitted by your calculator, compute $(1 + h)^{1/h}$, when $h = \pm 10^{-1}, \pm 10^{-2}, \pm 10^{-4}, \pm 10^{-6}$.

7.11. Prove that

(a) $(1 + h)^{1/h} > e$, if $-1 < h < 0$,

(b) $(1 + h)^{1/h} < e$, if $h > 0$.

7.12. Show $\lim_{n \to \infty}(1 + \frac{x}{n})^n = e^x$, for all x.

7.13. If $x > 1$, prove $\log(x + \sqrt{x^2 - 1}) = -\log(x - \sqrt{x^2 - 1})$.

7.14. Solve for x : $\int_2^x \frac{dt}{t} = 5 \int_1^x \frac{dt}{t}$.

7.15. Prove Napier's Inequality:

$$0 < a < b \Longrightarrow \frac{b-a}{b} < \log(\frac{b}{a}) < \frac{b-a}{a}.$$

7.16. Prove $\lim_{n \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \log 2$.

7.17. Use logarithic differentation to find $\frac{dy}{dx}$:

(a) $y = (x^3 + 2)^{1/2}(2 + x^2)^{1/3}$, (b) $y = \left(\frac{x^2 + 3}{x - 9}\right)^7$,

(c) $y = \sqrt{\frac{x^4-1}{x^4+1}}$,　　　　　　　　(d) $y = \frac{(x+2)^2(x+3)^3}{(x-1)(x+4)(x+5)^7}$.

7.18. Sketch the curves

(a) $y = \frac{\log x}{x}$, $x > 0$,　　(b) $y = x \log x$, $x > 0$.

7.19. Show $e \le a < b \implies a^b > b^a$.

7.20. Show that

(a) $\log(1 - e^{-x})$ is strictly increasing,

(b) $\frac{x}{e^x-1} - \log(1-e^{-x})$ is strictly decreasing, if $x > 0$. These
are the Einstein functions of radiation theory.

7.21. Show that f satisfies $f'(x) = f(x)$ for all $x \in \mathbb{R} \iff$
$f(x) = ce^x$. [*Hint:* Multiply $f'(x) - f(x) = 0$ by e^{-x} and
deduce that $e^{-x} f(x)$ is constant.]

7.22. If $b > 0, p, q \in \mathbb{R}$, show

(a) $b^p b^q = b^{p+q}$,　　(b) $(b^p)^q = b^{pq}$.

7.23. Let $b > 0, \alpha \in \mathbb{R}$. Prove

(a) $Db^x = b^x \log b$,　　(b) $Dx^\alpha = \alpha x^{\alpha-1}$,　$x > 0$.

7.24. Sketch the graphs of (a) e^{x^2},　(b) e^{-x^2},　(c) xe^{-x^2}.

§7.2 Trigonometric Functions.

The discussion of the trigonometric functions that follows does not conform to the standard of rigour which we have set ourselves. Such a discussion is well within our reach and a brief outline is given in the next section. However we prefer to concentrate initially on a careful description of the functions which adheres more closely to their geometric significance.

Consider the circle $x^2 + y^2 = 1$ centered at $O(0,0)$ and radius 1. The radian measure of the angle $\angle POA$ is t, the length of the arc AP described counterclockwise from A. Equivalently the radian measure of $\angle POA$ is 2 (area of the sector POA) $= 2(\frac{t}{2}) = t$. Define cosine and sine:
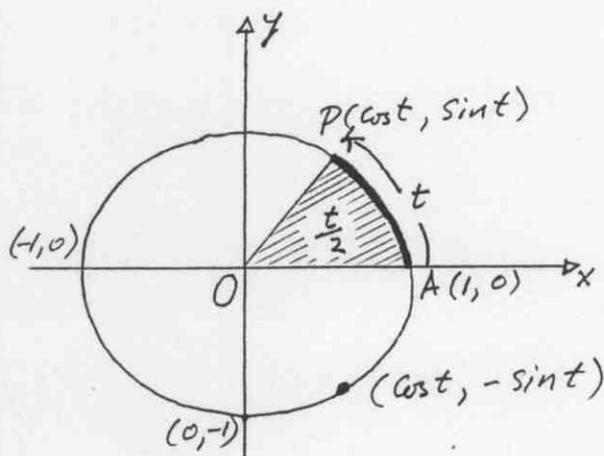
$$\cos t = x, \quad \sin t = y, \quad 0 \le t < 2\pi$$

where (x,y) are the coordinates of P. Notice that

$$\cos 0 = 1, \quad \sin 0 = 0, \quad \cos \frac{\pi}{2} = 0, \quad \sin \frac{\pi}{2} = 1,$$
$$\cos \pi = -1, \quad \sin \pi = 0, \quad \cos \frac{3\pi}{2} = 0, \quad \sin \frac{3\pi}{2} = -1.$$

The functions may now be extended to all real t by the periodicity conditions

$$\cos(t + 2\pi) = \cos t, \quad \sin(t + 2\pi) = \sin t$$

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which imply

$$\cos(t + 2n\pi) = \cos t, \quad \sin(t + 2n\pi) = \sin t, \quad n = 0, \pm 1, \pm 2, \dots.$$

Since the equation of the circle is $x^2 + y^2 = 1$, we have

(1) $$\cos^2 t + \sin^2 t = 1, \quad \text{for all} \quad t.$$

Since $(\cos t, -\sin t) = \big(\cos(2\pi - t), \sin(2\pi - t)\big) = \big(\cos(-t), \sin(-t)\big)$, by periodicity, so that

$$\cos t = \cos(-t), \quad \sin t = -\sin(-t);$$

cosine is an *even* function and *sine* is *odd*.

Observe that $d(A, P)$, the distance from A to P satisfies

$$d(A, P)^2 = (\cos t - 1)^2 + \sin^2 t = \cos^2 t - 2\cos t + 1 + \sin^2 t$$

$$= 2(1 - \cos t)$$

Therefore, if $P = (\cos t, \sin t)$, $Q = (\cos s, \sin s)$
$d(P, Q)^2 = 2[1 - \cos(t - s)]$.
On the other hand

$$d(P, Q)^2 = (\cos t - \cos s)^2 + (\sin t - \sin s)^2$$

$$= \cos^2 t - 2\cos t \cos s + \cos^2 s$$

$$+ \sin^2 t - 2\sin t \sin s + \sin^2 s$$

$$= 2[1 - (\cos t \cos s + \sin t \sin s)].$$
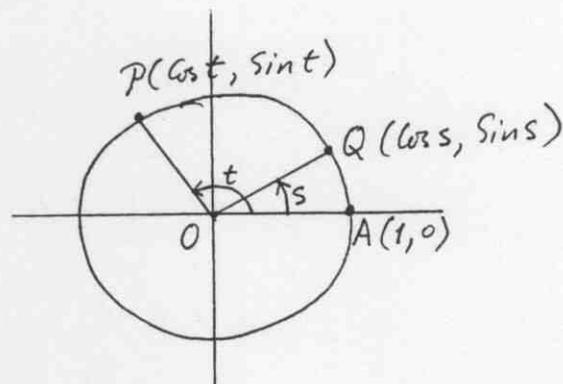
Thus it follows that

(2) $$\cos(t - s) = \cos t \cos s + \sin t \sin s.$$

The familiar trigonometric formulas all follow from (1), (2). Equation (2) implies

$$\cos(\frac{\pi}{2} - s) = \cos \frac{\pi}{2} \cos s + \sin \frac{\pi}{2} \sin s = \sin s,$$
$$\sin(\frac{\pi}{2} - s) = \cos[\frac{\pi}{2} - (\frac{\pi}{2} - s)] = \cos s.$$

Therefore

$$\sin(t - s) = \cos[\frac{\pi}{2} - (t - s)]$$
$$= \cos(\frac{\pi}{2} - t)\cos(-s) + \sin(\frac{\pi}{2} - t)\sin(-s)$$
$$= \sin t \cos s - \cos t \sin s;$$

(3) $$\sin(t - s) = \sin t \cos s - \cos t \sin s$$

Replacing s by $-s$ in (2), (3) gives

(4) $$\cos(t + s) = \cos t \cos s - \sin t \sin s$$

(5) $$\sin(t + s) = \sin t \cos s + \cos t \sin s$$

With $s = t$ in (4), (5) we obtain

(6) $$\cos 2t = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t = 2\cos^2 t - 1$$

(7) $$\sin 2t = 2\sin t \cos t.$$

Formula (6) also gives the important half-angle formulas

(8) $$\sin^2 t = \frac{1}{2}(1 - \cos 2t), \quad \cos^2 t = \frac{1}{2}(1 + \cos 2t).$$

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From (3) and (5) we find

$$\sin(t + s) + \sin(t - s) = 2\sin t \cos s$$

$$\sin(t + s) - \sin(t - s) = 2\cos t \sin s$$

or equivalently, with $\quad a = t + s, b = t - s$

$$(9) \qquad \sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$(10) \qquad \sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right).$$

Similarly, from (2) and (4)

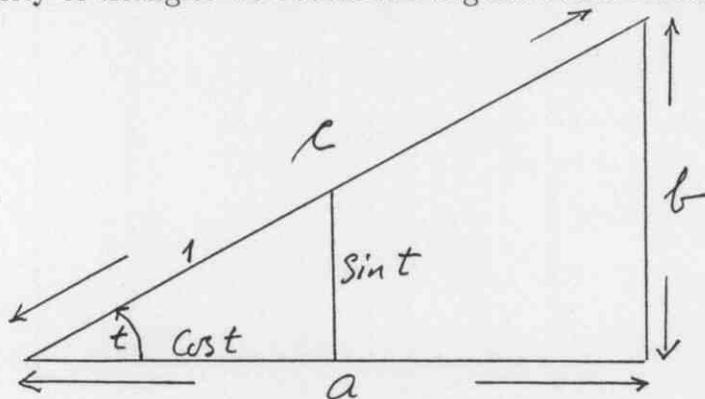$$(11) \qquad \cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$(12) \qquad \cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right).$$

The other standard trigonometric functions are

$$\tan t = \frac{\sin t}{\cos t}, \qquad \cot t = \frac{\cos t}{\sin t},$$

$$\sec t = \frac{1}{\cos t}, \qquad \operatorname{cosec} t = \frac{1}{\sin t}.$$

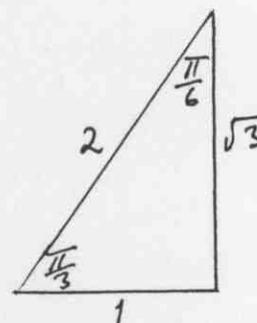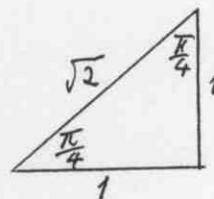From similarity of triangles we obtain the trigonometric functions as ratios.

$$\cos t = \frac{a}{c}, \quad \sin t = \frac{b}{c}, \quad \tan t = \frac{b}{a}.$$

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Some standard ratios are

$$\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

$$\cos \frac{\pi}{3} = \sin \frac{\pi}{6} = \frac{1}{2},$$

$$\sin \frac{\pi}{3} = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

To show $\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ first observe that

$$\cos \frac{\pi}{4} = \cos(\frac{\pi}{2} - \frac{\pi}{4}) = \sin \frac{\pi}{4} \quad \text{and so}$$

$$1 = \cos^2 \frac{\pi}{4} + \sin^2 \frac{\pi}{4} = 2 \cos^2 \frac{\pi}{2} \quad \text{and}$$

$$\frac{1}{\sqrt{2}} = \cos \frac{\pi}{4} = \sin \frac{\pi}{4}.$$

We must choose the positive square root since sine and cosine are both positive on $(0, \frac{\pi}{2})$.

To demonstrate the other four ratios we first see that $\frac{\pi}{2} = \frac{\pi}{6} = \frac{\pi}{3}$ implies

$$\cos \frac{\pi}{3} = \sin \frac{\pi}{6} \qquad \cos \frac{\pi}{6} = \sin \frac{\pi}{3}.$$

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Also $\frac{\pi}{3} = 2\frac{\pi}{6}$ implies

$$\sin \frac{\pi}{3} = \sin(2\frac{\pi}{6}) = 2\sin \frac{\pi}{6} \cos \frac{\pi}{6} \quad \text{from (7)}$$

$$= 2\sin \frac{\pi}{6} \sin \frac{\pi}{3}, \quad \text{and therefore}$$

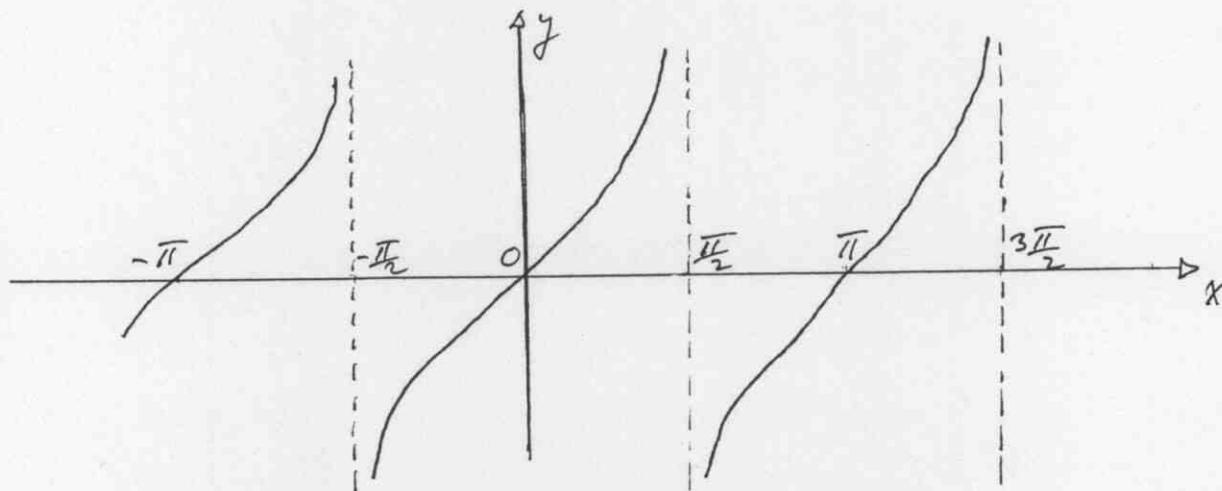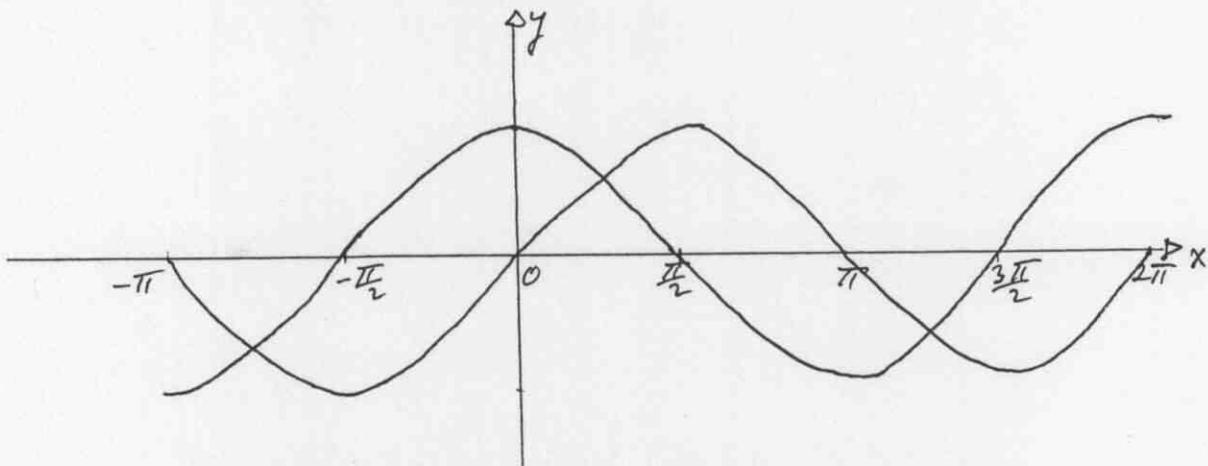$$1 = 2\sin \frac{\pi}{6}, \quad \text{since} \quad \sin \frac{\pi}{3} \neq 0, \quad \text{giving}$$

$$\frac{1}{2} = \sin \frac{\pi}{6} = \cos \frac{\pi}{3}.$$

Finally

$$1 = \cos^2 t + \sin^2 t, \quad t = \frac{\pi}{6}, \frac{\pi}{3}, \quad \text{gives}$$

$$\frac{\sqrt{3}}{2} = \cos \frac{\pi}{6} = \sin \frac{\pi}{3}.$$

Reasonable graphs of the trigonometric functions may be drawn directly from their description on the unit circle and the few explicit values we have found.

Now

$$\cos^2 t + \sin^2 t = 1 \Rightarrow$$

$$\cos^2 t \le 1, \quad \sin^2 t \le 1 \Rightarrow$$

(13) $$|\cos t| \le 1, \quad |\sin t| \le 1$$

If P is in the first quadrant, we have the following geometric

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inequality

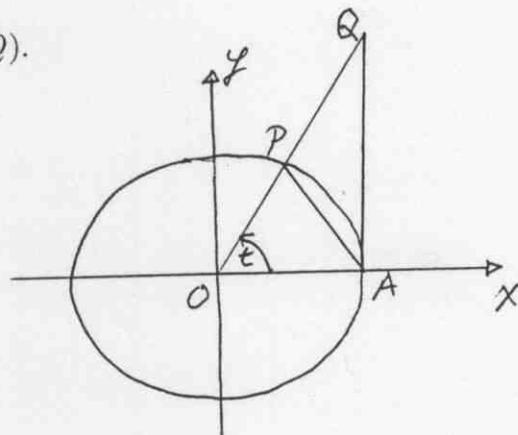area $(\triangle OAP) <$ area (sector OAP) < area $(\triangle OAQ)$.

Therefore

$\frac{1}{2}\sin t < \frac{1}{2}t < \frac{1}{2}\tan t$,   if $0 < t < \frac{\pi}{2}$.

Since the three functions involved are odd,

the inequalities are reversed if $\frac{-\pi}{2} < t < 0$.

Thus we get the two important inequalities

(14) $$|\sin t| < |t|, \quad 0 < |t| < \frac{\pi}{2}$$

(15) $$1 < \frac{t}{\sin t} < \frac{1}{\cos t}, \quad 0 < |t| < \frac{\pi}{2} .$$

Formula (12) implies

$$
\begin{aligned}
|\cos t - \cos a| &= |-2\sin(\frac{t+a}{2})\sin(\frac{t-a}{2})| \\
&= 2|\sin(\frac{t+a}{2})|\,|\sin(\frac{t-a}{2})| \\
&\leq 2 \cdot 1 \cdot |\frac{t-a}{2}| \\
&= |t-a|, \quad \text{from (13), (14)} \quad \Rightarrow
\end{aligned}
$$

(16) $$\lim_{t \to a} \cos t = \cos a.$$

Thus the cosine function is continuous on its domain. Since $\sin t = \cos(\frac{\pi}{2} - t)$, it follows that the sine function is also continuous on its domain.

(17) $$\lim_{t \to a} \sin t = \sin a.$$

Also, from (15), (16) (with $a = 0$)

(18) $$\lim_{t \to 0} \frac{\sin t}{t} = 1.$$

Finally, from (12)

$$\frac{[\cos(t + h) - \cos t]}{h} = -2\sin(\frac{2t + h}{2})\sin(\frac{h}{2})/h = -\sin(\frac{2t + h}{2}) \cdot \frac{\sin(\frac{h}{2})}{\frac{h}{2}}$$

and so from (17), (18) $lim_{h \to 0}[\cos(t + h) - \cos t]/h = -\sin t$

(19) $\quad D\cos t = -\sin t$

(20) $\quad D\sin t = \cos t, \quad$ since $\quad \sin t = \cos(\frac{\pi}{2} - t)$

(21) $\quad D\tan t = \sec^2 t, \quad$ since $\quad \tan t = \dfrac{\sin t}{\cos t}$

(22) $\quad D\cos t = -\operatorname{cosec}^2 t, \quad$ since $\quad \cot t = \dfrac{\cos t}{\sin t}$

(23) $\quad D\sec t = \sec t \tan t, \quad$ since $\quad \sec t = \dfrac{1}{\cos t}$

(24) $\quad D\operatorname{cosec} t = -\operatorname{cosec} t \cot t, \quad$ since $\quad \operatorname{cosec} t = \dfrac{1}{\sin t}.$
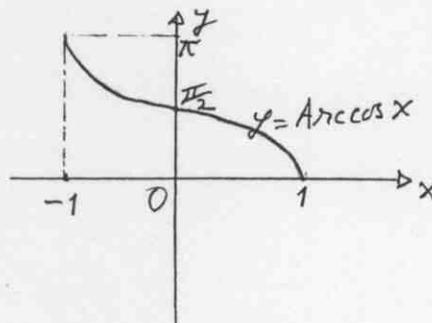
The cosine function is decreasing on $[0, \pi]$ and therefore, when restricted to this interval it has an inverse function, the Arccosine function with domain $[-1, 1]$ and range $[0, \pi]$ From Proposition 5.5.6 this function is differentiable on $(-1, 1)$ and

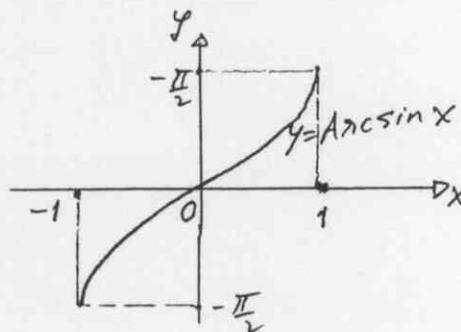(25) $$D \operatorname{Arccos} x = \frac{-1}{\sqrt{1 - x^2}}, \quad -1 < x < 1,$$

since

$$y = \operatorname{Arccos} x \Rightarrow \cos y = x \Rightarrow -(\sin y)\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1 - x^2}} \quad \text{(Why not} \quad \sin y = -\sqrt{1 - x^2}?).$$
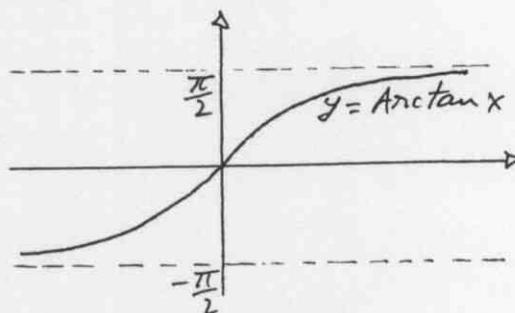
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Similarly the sine function is increasing on $[\frac{-\pi}{2}, \frac{\pi}{2}]$ and its restriction to this interval has an inverse function, Arcsine, where derivative on $(-1, 1)$ exists and

(26) $\qquad D \text{ Arcsin } x = \dfrac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$.

The tangent function is increasing on $(\frac{-\pi}{2}, \frac{\pi}{2})$ and its inverse, the Arctangent function satisfies

(27) $\qquad D \text{ Arctan } x - \dfrac{1}{1 + x^2}, \quad -\infty < x < \infty$.

§7.3. Trigonometric Functions, A Rigorous Approach.

This approach does not rely on concepts such as length of a curve which we have not yet considered. Also the pictures are used for motivation only and are not intrinsic to any proof. This section is based on Chapter 15 of 'Calculus' by M. Spivak.

DEFINITION 7.3.1. The number π

$$\pi = 2 \int_{-1}^{1} \sqrt{1 - u^2} \, du.$$

DEFINITION 7.3.2. The Arcosine function

$$\text{Arcos } x = x\sqrt{1 - x^2} + 2 \int_{x}^{1} \sqrt{1 - u^2} \, du, \quad -1 \le x \le 1.$$

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NOTE. The continuity of $\sqrt{1-x^2}$, $-1 \le x \le 1$, implies the existence of the integrals in these definitions. The motivation for the definitions is

(i) π is defined to be the area of the circular disc of radius 1.

(ii) The area of the sector OAP, with $x > 0$, is

$$\frac{t}{2} = \frac{1}{2} x\sqrt{1-x^2} + \int_x^1 \sqrt{1-u^2}\,du.$$

We call t the radian measure of $\angle POA$ and define $\text{Arcos}\,x = t$.

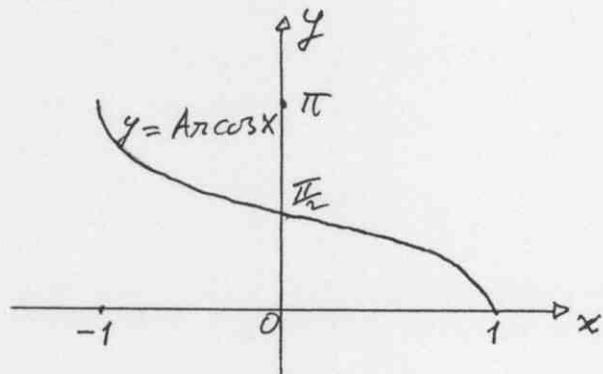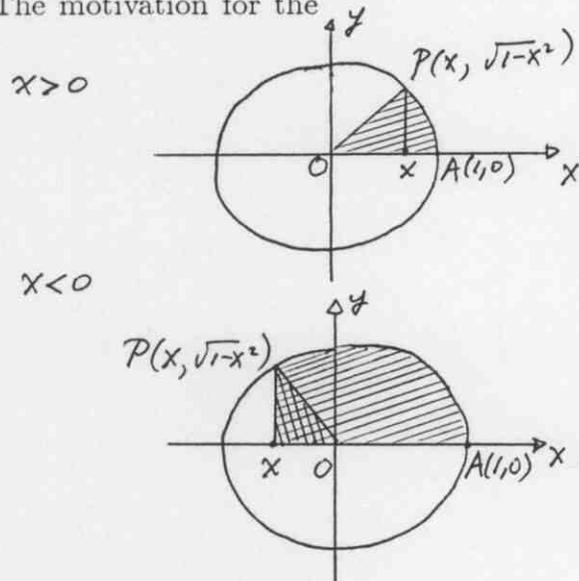The following are simple consequences of the definitions.

PROPOSITION 7.3.3.. $\text{Arcos}(-1) = \pi$, $\text{Arcos}\,0 = \frac{\pi}{2}$, $\text{Arcos}\,1 = 0$.

PROPOSITION 7.3.4..

$$D\,\text{Arcos}\,x = \frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1,$$

$$D^2\,\text{Arcos}\,x = \frac{-x}{(1-x^2)^{\frac{3}{2}}}, \quad -1 < x < 1.$$

We may now sketch the graph of the Arcosine function using its values at $-1, 0, 1$, the fact that it is decreasing on $[-1, 1]$, concave up on $[-1, 0]$ and concave down on $[0, 1]$.
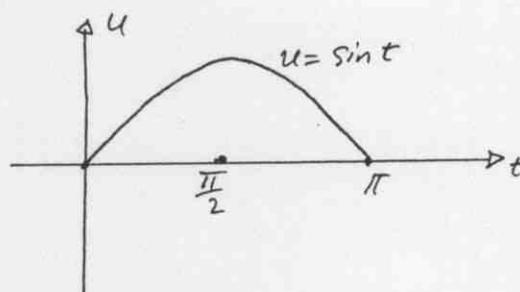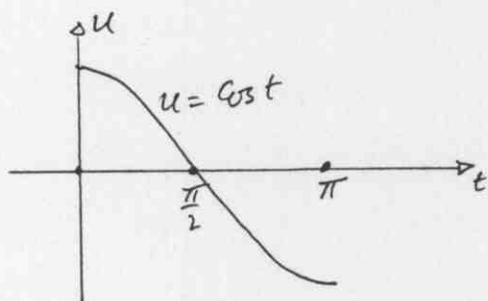
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DEFINITION 7.3.5.

(a) On $[0, \pi]$ cosine is defined to be the inverse function of

Arcosine

$$-1 \leq x \leq 1$$
$$t = \text{Arcos } x \iff \cos t = x,$$
$$0 \leq t \leq \pi.$$

(b) On $[0, \pi]$ the sine function is defined by

$$\sin t = \sqrt{1 - \cos^2 t} = \sqrt{1 - x^2}, \quad 0 \leq t \leq \pi.$$

PROPOSITION 7.3.6. $\cos^2 t + \sin^2 t = 1 \quad 0 \leq t \leq \pi$

PROPOSITION 7.3.7.

$$D \cos t = - \sin t,$$

$$D \sin t = \cos t, \quad 0 \leq t \leq \pi$$

PROOF: From Proposition 5.5.6, the cosine function has a derivative.

Therefore $x = \cos t, \quad 0 \leq t \leq \pi$

$$\Rightarrow \text{Arcos } x = t$$

$$\Rightarrow \frac{-1}{\sqrt{1 - x^2}} \cdot \frac{dx}{dt} = 1 \quad \text{(Proposition 7.3.4, Chain Rule)}$$

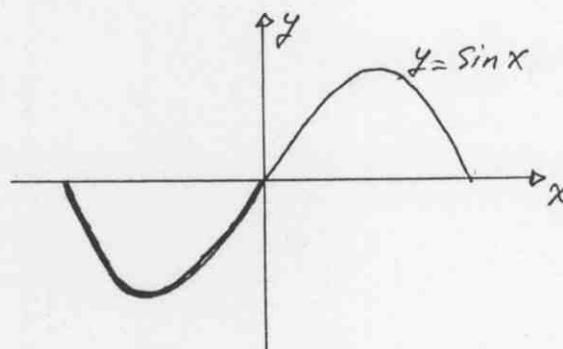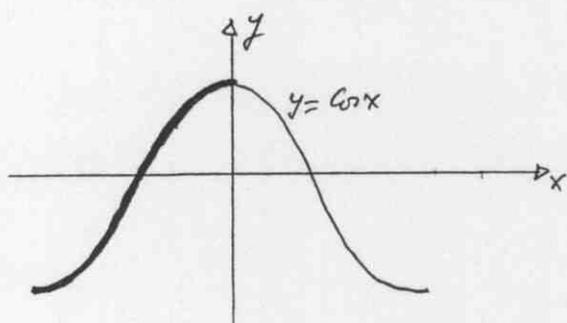$$\Rightarrow \frac{dx}{dt} = -\sqrt{1 - x^2} = - \sin t.$$

Also

$$y = \sin t = \sqrt{1 - \cos^2 t}$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{2\sqrt{1 - \cos^2 t}}(-2\cos t)(-\sin t) = \cos t.$$

$\square$

Now the domains of cosine and sine may be extended to all of $\mathbb{R}$ in two steps. First we extend them to $[-\pi, \pi]$ by

$$\cos(-x) = \cos x \qquad\qquad \sin(-x) = -\sin x$$

cosine is *even*   sine is *odd*

The functions are then extended to $\mathbb{R}$ by the periodicity

$$\cos(x + 2n\pi) = \cos x$$

$$\sin(x + 2n\pi) = \sin x \qquad n = 0, \pm 1, \pm 2, \ldots .$$

It is fairly easy to see that Propositions 7.3.6, 7.3.7. extend to all $t \in \mathbb{R}$ also.

The formula $\cos(t - s) = \cos t \cos s + \sin t \sin s$ was deduced from the geometry of the circle and was shown to imply all of the other trigonometric formulas including those fundamental to finding differentiation formulas in the approach developed in the preceding

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section. Here we already have the differentiation formulas and now show how they imply the trigonometric identities.

LEMMA 7.3.8.. *Suppose* f *is a twice differentiable function on* $\mathbb{R}$ *which satisfies*

$$f(0) = f'(0) = 0 \quad \text{and} \quad f'' + f = 0 \quad \text{on} \quad \mathbb{R}.$$

Then $f(t) = 0$ for all $t \in \mathbb{R}$

PROOF: Let $g = (f')^2 + f^2$. Then

$$g' = 2f'f'' + 2ff' = 2f'(f'' + f) = 0 \quad \text{on} \quad \mathbb{R}$$
$$\Rightarrow g = \text{constant} = g(0) = 0 \Rightarrow f = 0.$$

$\square$

PROPOSITION 7.3.9. *For all* $t, s \in \mathbb{R}$

$$\cos(t - s) = \cos t \cos s + \sin t \sin s.$$

PROOF: Consider

$$f(t) = \cos(t - s) - a \cos t - b \sin t, \quad a, b, s \quad \text{constants}$$
$$f'(t) = -\sin(t - s) + a \sin t - b \cos t$$
$$f''(t) = -\cos(t - s) + a \cos t + b \sin t.$$

Therefore $f''(t) + f(t) = 0$ for all a, b, s. Now choose a, b so that $0 = f(0) = f'(0)$ and we find $f(t) = 0$ for all t from

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Lemma 7.3.8.

$$0 = f(0) = \cos(-s) - a \Rightarrow a = \cos(-s) = \cos\ s$$

$$0 = f'(0) = -\sin(-s) - b \Rightarrow b = -\sin(-s) = \sin\ s$$

$$\Rightarrow\ 0 = f(t) = \cos(t - s) - \cos\ s\ \cos\ t - \sin\ s\ \sin\ t, \quad \text{all}$$

$t \in \mathbb{R}$. Since s is arbitrary, this completes the proof.

$$\square$$

In §7.2 we needed to know that $\lim_{t \to 0} \frac{\sin\ t}{t} = 1$ to drive the formulas $D \sin\ t = \cos\ t$, $D \cos\ t = -\sin\ t$. We therefore could not use l'Hospital's Rule as a proof of this limit. In the present approach we may use l'Hospital's Rule since we did not need the result in our proof of Proposition 7.3.7. Therefore

$$\lim_{t \to 0} \frac{\sin\ t}{t} = \lim_{t \to 0} \frac{\cos\ t}{1} = 1, \quad \text{by l'Hospital's Rule.}$$

§7.4. Hyperbolic Functions.

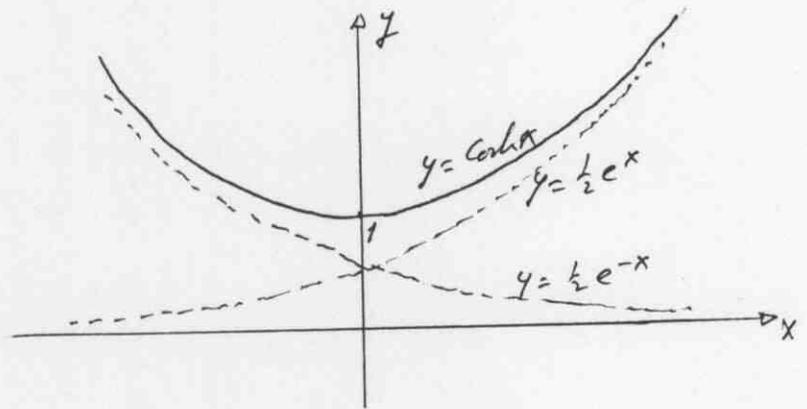The three basic functions are the hyperbolic cosine, hyperbolic sine and hyperbolic tangent.

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x})$$

$$\cosh(-x) = \cosh x$$
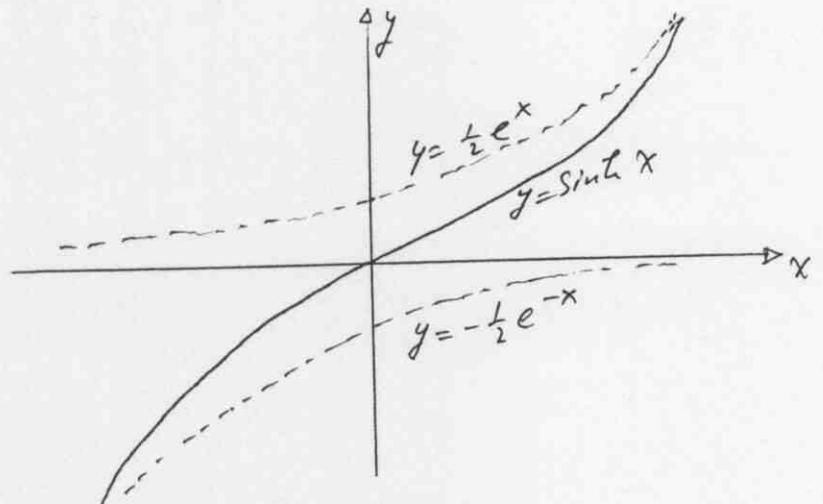
an *even* function

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x})$$
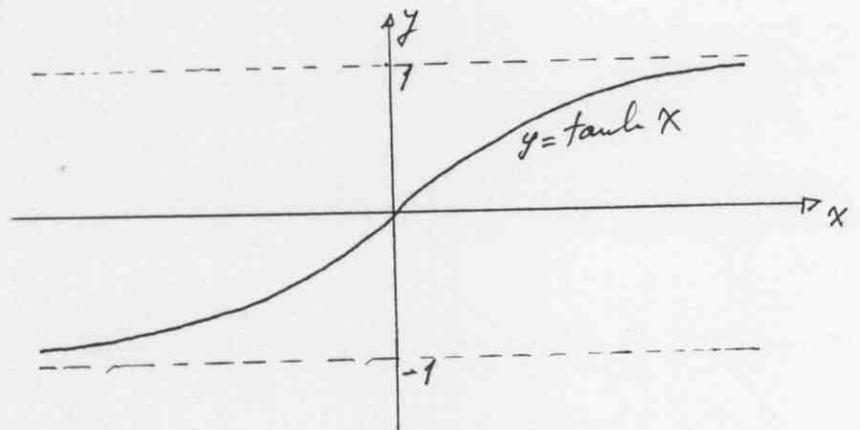
$$\sinh(-x) = -\sinh x$$

an *odd* function

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$= \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\tanh(-x) = -\tanh x$$

an *odd* function.

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Other hyperbolic functions are

$$\coth x = \frac{1}{\tanh x}, \quad \operatorname{csch} x = \frac{1}{\sinh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x}.$$

We have the following differentation formulas:

$$D \cosh x = \sinh x, \quad D \sinh x = \cosh x, \qquad D \tanh x = \operatorname{sech}^2 x$$

$$D \coth x = -\operatorname{csch}^2 x, \quad D \operatorname{sech} x = -\operatorname{sech} x \tanh x, \quad D \operatorname{csch} x = -\operatorname{csch} x \coth x.$$

In the same way that the trigonometric functions are associated with the circles, the hyperbolic functions are associated with the hyperbola.

$$\cosh^2 x - \sinh^2 x = 1.$$

This can be seen directly from the definition or from $D(\cosh^2 x - \sinh^2 x) = 2 \cosh x \sinh x - 2 \sinh x \cosh x = 0$. Which implies $\cosh^2 x - \sinh^2 x = \text{constant} = \cosh^2 0 - \sinh^2 0 = 1$. Thus

$$x = \cosh t, \quad y = \sinh t \Rightarrow x^2 - y^2 = 1.$$

We will see later that

the sector OAP has area $\frac{1}{2} |t|$. (Problem 9.5)

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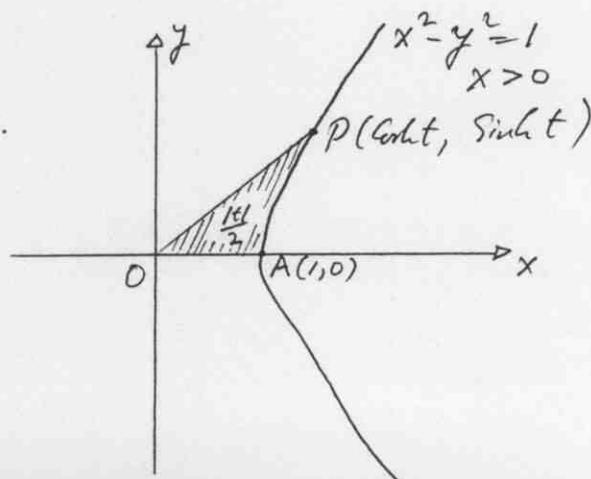
The formulas

$$\sinh(x \pm y) = \sinh x \ \cosh y \pm \cosh x \ \sinh y$$

$$\cosh(x \pm y) = \cosh x \ \cosh y \pm \sinh x \ \sinh y$$

may be proved from the definition by first proving the expression for one, $\sinh(x+y)$ for example, then replacing y by $-y$ to obtain $\sinh(x-y)$. These formulas may then be differentiated with respect to x to obtain the formulas for $\cosh(x+y), \cosh(x-y)$.

Problems

7.25. Show $\quad 1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x.$

7.26. Show (a) $\quad \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta},$

(b) $\quad \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}.$

7.27. Show (a) $\quad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0,$

(b) $\quad \lim_{x \to 0} \frac{\cos x - 1}{x^2} = \frac{-1}{2}.$

7.28. Sketch the curve $\quad y = \cos 2x + 2 \cos x, \quad 0 \le x \le 2\pi.$

7.29. Express as multiples of π :

(a) $\text{Arcos}(\frac{1}{\sqrt{2}}),$ (b) $\text{Arcsin}(\frac{\sqrt{3}}{2}),$ (c) $\text{Arcos}(\frac{1}{2}),$

(d) $\text{Arctan } \sqrt{3},$ (e) $\text{Arctan}(-\frac{1}{\sqrt{3}}),$ (f) $\text{Arctan}(-1).$

7.30. Write down the value of

(a) $\sin(\text{Arctan } 5),$ (b) $\tan(\text{Arcsin } \frac{1}{3}),$ (c) $\log(e^2).$

7.31. Show (a) $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{6},$ (b) $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}.$

7.32. Find $\frac{dy}{dx}$:

(a) $y = \frac{x \tan x}{1+x^2},$ (b) $y = \text{Arcsin } \sqrt{x},$

(c) $y = \text{Arcsin } \sqrt{1-x^2},$ (d) $y = \text{Arcsin } x + \sqrt{1-x^2},$

(e) $y = \text{Arctan}(x + \sqrt{1+x^2}),$ (f) $y = \sin(\text{Arctan } x)$

(g) $y = e^{(\sin x)^2},$ (h) $y = e^{\sin(x^2)},$

(i) $y = \sin(\log x).$

7.33. Show $\text{Arctan}\left(\frac{x}{\sqrt{1-x^2}}\right) = \text{Arcsin}\ x,\quad -1 < x < 1.$

7.34. If $\text{Arctan}\left(\frac{x}{y}\right) + \log\ \sqrt{x^2+y^2} = 0,$ show that $\frac{dy}{dx} = \frac{x+y}{x-y}.$

7.35 Prove $\text{Arctan}\ x + \text{Arctan}\ y = \text{Arctan}\left(\frac{x+y}{1-xy}\right)$ if $|\text{Arctan}\ x +$ $\text{Arctan}\ y| < \frac{\pi}{2}.$

7.36. Prove $\text{Arcsin}\ x + \text{Arcsin}\ y = \text{Arcsin}(x\ \sqrt{1-y^2} + y\ \sqrt{1-x^2})$ for certain $x, y.$

7.37. By restricting the domain of the hyperbolic function if necessary define the inverse functions $\cosh^{-1}, \sinh^{-1}, \tanh^{-1}$ and show

(a) $D\ \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}},$ \qquad (b) $D\ \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}},$

(c) $D\ \tanh^{-1} x = \frac{1}{1-x^2},$ and state the values of x for which each formula is valid.

7.38. Show

(a) $\cosh^{-1} x = \log(x + \sqrt{x^2-1}),$

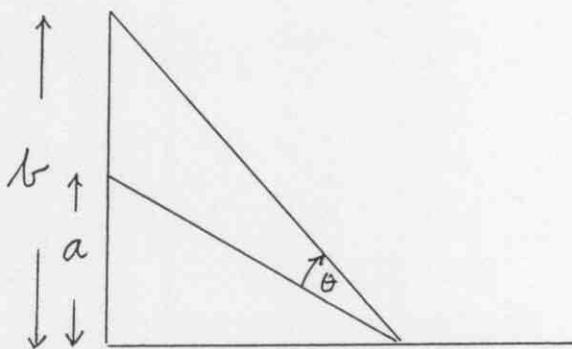(b) $\sinh^{-1} x = \log(x + \sqrt{x^2+1})$

(c) $\tanh^{-1} x = \log\ \sqrt{\frac{1+x}{1-x}}.$

7.39. If π and $\text{Arcos}\ x$ are given by the Definitions 7.3.1 - 7.3.2, show

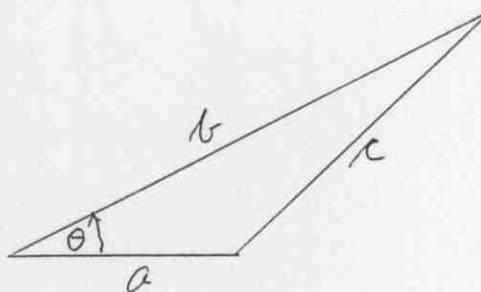(a) $\text{Arcos}(-1) = \pi,\quad \text{Arcos}\ 0 = \frac{\pi}{2},\quad \text{Arcos}\ 1 = 0.$

(b) $D\ \text{Arcos}\ x = \frac{-1}{\sqrt{1-x^2}},\quad -1 < x < 1.$

7.40. Show that the

largest value of the

angle θ in the

diagram is

arctan $\left(\frac{b-a}{2\sqrt{ab}}\right)$.

7.41. Prove the *Cosine Law:*

If a, b, c are the sides of

a triangle and θ is the

angle between $a, b,$ then

$c^2 = a^2 + b^2 - 2ab \cos \theta$.

7.42. Show

(a) $\sin t \cos s = \frac{1}{2}[\sin(t+s) + \sin(t-s)]$.

HINT: Use (3), (5), of § 7.2.

(b) $\cos t \cos s = \frac{1}{2}[\cos(t+s) + \cos(t-s)]$. Find a similar

formula for $\sin t \sin s$.

7.43. On a clock, the minute and hour hands have length a cm and

b cm respectively. Find the rate at which the distance between

the tips of the hands is

(a) decreasing at 3:00,

(b) increasing at 8:00.

VIII TECHNIQUES OF INTEGRATION

§8.1. The Fundamental Theorem of Calculus.

We prove a slightly stronger form of this theorem than Theorem 6.4.4.

THEOREM 8.1.1.. *Suppose (i)* $\int_a^b f$ *exists and*

(ii)
$$F'(x) = f(x), \quad a \leq x \leq b.$$

Then

$$\int_a^b f = F(b) - F(a).$$

PROOF: Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ i.e. $a = x_0 < x_1 < \cdots < x_n = b$.

$$F(b) - F(a) = \sum_{k=1}^{n} [F(x_k) - F(x_{k-1})]$$

$$= \sum_{k=1}^{n} F'(c_k)(x_k - x_{k-1}), \quad \text{for some} \quad c_k \in (x_{k-1}, x_k) \quad \text{(Why?)}$$

$$= \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}), \quad \text{by (ii).}$$

Therefore $L(P, f) \leq F(b) - F(a) \leq U(P, f)$ for each partition P of $[a, b]$, so that

$$\int_a^b f = F(b) - F(a),$$

since $\int_a^b f$ exists from (i).

$\square$

NOTES:

(i) $\int_a^b f$, $\int_a^b f(x)dx$, $\int_a^b f(t)dt$, $\int_a^b f(u)du$ all mean the same thing.

(ii) F is called an *antiderivative* of f.

(iii) If F is an antiderivative of f, then G is also an antiderivative of $f \iff F(x) = G(x) + c$, c constant, $a \leq x \leq b$.

(iv) In Theorem 6.4.4. it is assumed that f is continuous on $[a, b]$ which implies (i) and that $F(x) = \int_a^x f$ satisfies (ii). Thus Theorem 8.1.1 $\Rightarrow$ Theorem 6.4.4.

(v) All of the techniques developed in this chapter are methods of using the fundamental theorem.

(vi) The notation $\int f(x)dx = F(x)$ means 'F is an antiderivative of f'. Of course F is only determined to within a constant. The following are some useful antiderivatives:

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha + 1} + c, \quad \alpha \neq -1.$$

$$\int x^{-1} dx = \log|x| + c = \log k|x| \quad (c = \log k)$$

$$\int e^x dx = e^x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \sin x \, dx = -\cos x + c$$

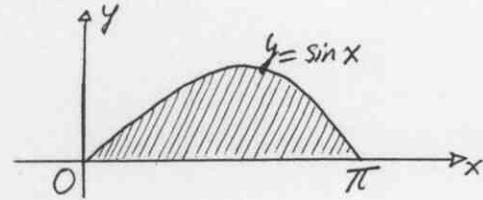$$\int \frac{1}{\sqrt{1 - x^2}} dx = \text{Arcsin } x + c = -\text{Arcos } x + k$$

$$\int \frac{1}{1 + x^2} dx = \text{Arctan } x + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

$$\int \csc^2 x \, dx = -\cot x + c.$$

This list may be greatly extended by the Change of Variable or Substitution Formula of the next section.

EXAMPLE 8.1.2: The area determined by an arch of the sine curve is

$$\int_0^\pi \sin x \, dx = -\cos x \mid_0^\pi = -\cos \pi + \cos 0 = -(-1) + 1 = 2.$$

EXAMPLE 8.1.3:

$$\int_0^{\sqrt{3}} \frac{1}{1+x^2} \, dx = \arctan x \mid_0^{\sqrt{3}} = \arctan \sqrt{3} - \arctan 0 = \frac{\pi}{3}.$$

§8.2 The Change of Variable or Substitution Formula.

This is really a combination of the Fundamental Theorem of Calculus and the Chain Rule.

THEOREM 8.2.1.. *Suppose g' is continuous on $[a, b]$ and f is continuous on $g([a, b])$. Then*

(i) $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$, *or equivalently*

(ii) $\int f(g(x))g'(x)dx = \int f(u)du$, *where $u = g(x)$.*

PROOF: Let F be an antiderivative of f, i.e. $F'(u) = f(u)$ for each $u \in g([a, b])$. Consider also $H(x) = F(g(x))$. Then $H'(x) = F'(g(x))g'(x) = f(g(x))g'(x)$ so that H is an antideriva-

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tive of $(f \circ g)g'$. Therefore

$$\int_a^b f(g(x))g'(x)dx = H(x)|_a^b = F(g(x))|_a^b = F(g(b)) - F(g(a))$$

$$\int_{g(a)}^{g(b)} f(u)du = F(u)|_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

which gives us (i). Similarly

$$\int f(g(x))g'(x)dx = H(x) + c = F(g(x)) + c$$

$$\int f(u)du = F(u) + c = F(g(x)) + c, \quad \text{since} \quad u = g(x)$$

gives the form (ii) of this theorem.

$\square$

NOTES:

(i) In practice the choice of the function g may not be obvious and there may be many different choices of g which do work.

(ii) The notation 'dx', 'du' is very convenient here. When we make the substitution

$$u = g(x)$$

we take

$$du = g'(x)dx$$

and also change the integration interval: $u = g(a), g(b)$ when $x = a, b$ respectively.

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EXAMPLE 8.2.2:

$$\int_1^3 10(x^2+5)^{27}x\,dx \qquad u = x^2+5$$

$$du = 2x\,dx$$

$$= \int_6^{14} 5u^{27}\,du \qquad x=1,\ u=6$$

$$= \frac{5}{28}u^{28}\big|_6^{14} = \frac{5}{28}(14^{28}-6^{28}) \qquad x=3,\ u=14$$

Of course you could also expand $(x^2+5)^{27}$ multiply by $10x$ and carry out the resulting 28 integrations.

EXAMPLE 8.2.3:

$$\int \tan x\,dx = \log|\sec x| + c$$

since

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx \qquad \begin{matrix} u = \cos x \\ du = -\sin x\,dx \end{matrix}$$

$$= -\int \frac{du}{u}$$

$$= -\log|u| + c = -\log|\cos x| + c$$

$$= \log|\sec x| + c.$$

EXAMPLE 8.2.4:

$$\int_1^e \frac{(\log x)^2}{x}\,dx \qquad \begin{matrix} u = \log x, & du = \frac{1}{x}\,dx \\ x = 1, u = 0 \\ x = e, & u = 1 \end{matrix}$$

$$= \int_0^1 u^2\,du$$

$$= \frac{u^3}{3}\big|_0^1 = \frac{1}{3}.$$

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EXAMPLE 8.2.5:

$$\int \frac{e^t}{e^t + 1}\, dt \qquad u = e^t + 1,\ du = e^t dt$$

$$= \int \frac{1}{u}\, du$$

$$= \log |u| + c = \log(e^t + 1) + c = \log[k(e^t + 1)].$$

EXAMPLE 8.2.6:

$$\int e^{x^2} x\, dx = \frac{1}{2} \int e^u du \quad u = x^2,\ du = 2x dx$$

$$= \frac{1}{2}\, e^u + c = \frac{1}{2}\, e^{x^2} + c.$$

EXAMPLE 8.2.7:

$$\int \frac{x\, dx}{\sqrt{3x + 5}} \qquad \begin{array}{ll} u = 3x + 5 & du = 3dx \\ x = \frac{1}{3}(u - 5) & \frac{1}{3}\, du = dx \end{array}$$

$$= \int \frac{\frac{1}{3}(u - 5)\frac{1}{3} du}{\sqrt{u}}$$

$$= \frac{1}{9} \int \frac{u - 5}{\sqrt{u}} du = \frac{1}{9} \int (u^{\frac{1}{2}} - 5u^{-\frac{1}{2}}) du$$

$$= \frac{1}{9}\Big(\frac{2}{3}\, u^{\frac{2}{3}} - 5.2u^{\frac{1}{2}}\Big) + c$$

$$= \frac{2}{9}[\frac{1}{3}(3x + 5)^{\frac{3}{2}} - 5(3x + 5)^{\frac{1}{2}}] + c$$

$$= \frac{2}{9}(3x + 5)^{\frac{1}{2}}(x - \frac{10}{3}) + c.$$

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Alternative.

$$\int \frac{x\,dx}{3x+5}$$

$$v = \sqrt{3x+5}$$
$$v^2 = 3x + 5 \quad 2v\,dv = 3dx$$

$$= \tfrac{2}{9} \int (v^2 - 5)dv$$
$$\tfrac{1}{3}(v^2 - 5) = x$$
$$= \tfrac{2}{9}(\tfrac{1}{3}\,v^3 - 5v) + c$$
$$\tfrac{2}{9}(v^2 - 5)dv = \frac{x\,dx}{\sqrt{3x+5}}$$
$$= \tfrac{2}{9}[\tfrac{1}{3}(3x+5)^{\frac{3}{2}} - 5(3x+5)^{\frac{1}{2}}] + c$$
$$= \tfrac{2}{9}(3x+5)^{\frac{1}{2}}(x - \tfrac{10}{3}) + c.$$

It may be helpful to break the integral up and use different techniques on different portions

EXAMPLE 8.2.8:

$$\int \frac{8x - 7}{x^2 + 1}dx = 4 \int \frac{2x\,dx}{x^2 + 1} - 7 \int \frac{dx}{x^2 + 1}$$
$$= 4\log(x^2 + 1) - 7\arctan x + c.$$

§8.3. Special Substitutions

$$\int f(\cos x)\sin x\,dx = -\int f(u)du, \quad \text{where} \quad u = \cos x$$
$$\int f(\sin x)\cos x\,dx = \int f(u)du, \quad \text{where} \quad u = \sin x.$$

These substitutions together with the identity $\cos^2 x + \sin^2 x = 1$ may be used to evaluate integrals of the form

$$\int \cos^m x \, \sin^n x \, dx$$

if one of m, n is an odd integer.

If n is odd, use $u = \cos x$.

If m is odd, use $u = \sin x$.

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EXAMPLE 8.3.1:

$$\int \cos^{\frac{1}{5}} x \, \sin^3 x \, dx = \int \cos^{\frac{1}{5}} x (1 - \cos^2 x) \sin x \, dx$$

$$= -\int (u^{\frac{1}{5}} - u^{\frac{11}{5}}) du, \quad \text{where} \quad u = \cos x, du = \sin x \, dx$$

$$= -(\frac{5}{6} u^{\frac{6}{5}} - \frac{5}{16} u^{\frac{16}{5}}) + c$$

$$= 5 \cos^{\frac{6}{5}} x (\frac{1}{16} \cos^2 x - \frac{1}{6}) + c.$$

EXAMPLE 8.3.2:

$$\int \cos^3 2t \, dt = \int (1 - \sin^2 2t) \cos 2t \, dt$$

$$= \frac{1}{2} \int (1 - u^2) du \quad \text{where} \quad u = \sin 2t \quad du = 2 \cos 2t \, dt$$

$$= \frac{1}{2}(u - \frac{u^3}{3}) + c$$

$$= \frac{1}{2} \sin 2t - \frac{1}{6} \sin^3 2t + c.$$

When m, n are both even integers, then the trigonometric identities

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x), \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

may be used (repeatedly if necessary) to reduce the problem to a situation where m is odd.

EXAMPLE 8.3.3:

$$\int \sin^2 x \, dx = \frac{1}{2} \int (1 - \cos 2x) dx$$

$$= \frac{1}{2} x - \frac{1}{4} \sin 2x + c.$$

EXAMPLE 8.3.4:

$$\int \cos^4 x \, dx = \int (\cos^2 x)^2 dx$$

$$= \frac{1}{4} \int (1 + \cos 2x)^2 dx$$

$$= \frac{1}{4} \int (1 + 2\cos 2x + \cos^2 2x) dx$$

$$= \frac{1}{4} \int [1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)] dx$$

$$= \frac{1}{4} \int (\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x) dx$$

$$= \frac{1}{4}(\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x) + c.$$

Trigonometric Substitution. When the integrand contains expressions of the form

$$a^2 - u^2; \quad a^2 + u^2; \quad u^2 - a^2$$

and no more obvious techniques is available, then it is worth trying the substitution

$$u = a \sin \theta; \quad u = a \tan \theta; \quad u = a \sec \theta$$

respectively. Then

$$a^2 - u^2 = a^2 \cos^2 \theta; \quad a^2 + u^2 = a^2 \sec^2 \theta; \quad u^2 - a^2 = a^2 \tan^2 \theta.$$

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EXAMPLE 8.3.5:

$$\int \frac{du}{\sqrt{a^2 - u^2}}, \ a > 0. \quad \text{Let } u = a \sin \theta, \quad du = a \cos \theta \ d\theta$$

$$a^2 - u^2 = a^2 \cos^2 \theta, \quad \sqrt{a^2 - u^2} = a \cos \theta^*$$

$$= \int \frac{a \cos \theta \ d\theta}{a \cos \theta}$$

$$= \int d\theta = \theta + c \quad = \arcsin(\tfrac{u}{a}) + c.$$

*Why was ' $- a \cos \theta$ ' not chosen here?

EXAMPLE 8.3.6:

$$\int \frac{du}{a^2 + u^2}, \ a > 0 \quad \text{Let } \ u = a \tan \theta, \ du = a \sec^2 \theta \ d\theta$$

$$a^2 + u^2 = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta$$

$$= \int \frac{a \sec^2 \theta \ d\theta}{a^2 \sec^2 \theta}$$

$$= \tfrac{1}{a} \int d\theta \qquad = \tfrac{1}{a} \ \theta + c = \tfrac{1}{a} \arctan(\tfrac{u}{a}) + c.$$

EXAMPLE 8.3.7:

$$\int \frac{du}{\sqrt{u^2 - a^2}}, \ a > 0. \qquad \text{Let } \ u = a \sec \theta, \quad du = a \sec \theta \tan \theta$$

$$u^2 - a^2 = a^2(\sec^2 \theta - 1) = a^2 \tan^2 \theta$$

$$= \pm \int \frac{a \sec \theta \tan \theta \ d\theta}{a \tan \theta} \quad \sqrt{u^2 - a^2} = a|\tan \theta| = \pm a \tan \theta$$

$$= \pm \int \sec \theta \ d\theta \qquad \begin{array}{l} \text{' + ' if} \quad \tan \theta \geq 0 \\ \text{' - ' if} \quad \tan \theta < 0 \end{array}$$

We don't have an antiderivative for 'sec' yet (see Problem 8.3). Try $\int \sec \theta \ d\theta = \int \frac{d\theta}{\cos \theta}$. (This is the 'm odd' situation just discussed.)

$$= \int \frac{\cos \theta}{\cos^2 \theta} \ d\theta = \int \frac{\cos \theta \ d\theta}{1 - \sin^2 \theta} \qquad v = \sin \theta dv = \cos \theta \ d\theta$$

$$= \int \frac{dv}{1 - v^2} \ .$$

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We're still in trouble, but we will be able to handle this shortly when we discuss antidifferentation of rational functions by the method of partial fractions.

The forms $a^2 - u^2, a^2 + u^2, u^2 - a^2$ of this discussion may not always be entirely obvious.

EXAMPLE 8.3.8:

$$\int \frac{dx}{5 + 2x + x^2} = \int \frac{dx}{4 + (x+1)^2} \qquad \text{Let} \quad x + 1 = 2\tan\theta$$

$$dx = 2\sec^2\theta \, d\theta$$

$$= \int \frac{2\sec^2\theta}{4\sec^2\theta} \, d\theta \qquad\qquad 4 + (x+1)^2 = 4\sec^2\theta$$

$$= \tfrac{1}{2} \int d\theta$$

$$= \tfrac{1}{2}\theta + c = \tfrac{1}{2}\arctan(\tfrac{x+1}{2}) + c.$$

Problems

8.1 Find the following antiderivatives:

(a) $\int \sin^5 x \cos x \, dx$,

(b) $\int e^{x^2} x \, dx$,

(c) $\int \frac{1}{\sqrt{x}} \sin \sqrt{x} \, dx$,

(d) $\int \frac{dt}{(1+t^2)\arctan t}$,

(e) $\int \frac{ds}{\sqrt{5s+3}}$, (f) $\int \tanh u \, du$.

ANS: (a) $\frac{1}{6} \sin^6 x + c$, (c) $-2\cos \sqrt{x} + c$

(e) $\frac{2}{5} \sqrt{5s+3} + c$.

8.2 Find the antiderivatives:

(a) $\int \frac{e^x \, dx}{1+e^{2x}}$

(b) $\int \sqrt{1-x^2} \, dx$

(c) $\int (4+u^2)^{\frac{-3}{2}} du$

(d) $\int \frac{x}{\sqrt{1-x^2}} \, dx$

(e) $\int \frac{dt}{1+(5t+3)^2}$

(f) $\int \frac{x}{\sqrt{1-x^4}} \, dx$.

ANS: (a) $\arctan(e^x) + c$, (c) $\frac{1}{4} \frac{u}{(4+u^2)^{1/2}} + c$,

(e) $\frac{1}{5} \arctan(5t+3) + c$.

8.3 Verify the formula:

$$\int \sec \theta d\theta = \log |\sec \theta + \tan \theta| + c.$$

8.4 Find the antiderivative $\int \sin \theta \cos \theta \, d\theta$ in 3 ways

(a) by using the substitution $u = \cos \theta$,

(b) by using $u = \sin \theta$

(c) by using the identity $\sin 2t = 2 \sin t \cos t$.

Reconcile the answers you obtain.

§8.4. Integration by Parts.

Just as the Chain Rule combines with the Fundamental Theorem of Calculus to give us The Substitution Formula in Theorem 8.2.1, the Product Formula leads to another important technique, the Integration by Parts Formula.

THEOREM 8.4.1.. *Suppose* f', g' *are continuous on* $[a, b]$. *Then*

(i) $\int_a^b f(x)g'(x)dx = f(x)g(x)\big|_a^b - \int_a^b g(x)f'(x)dx$, *or equivalently*

(ii) $\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$.

PROOF: If $F = fg$, then $F' = fg' + gf'$ and $\int_a^b F'(x)dx = F(b) - F(a) = F(x)\big|_a^b$ implies

$$\int_a^b [f(x)g'(x) + g(x)f'(x)]dx = f(x)g(x)\big|_a^b.$$

$\square$

NOTES:

(i) A convenient shorthand for the Integration by Parts Formula is obtained by taking

$$u = f(x) \qquad v = g(x)$$

$$du = f'(x)dx \qquad dv = g'(x)dx$$

and then:

$$\int u\, dv = uv - \int v\, du.$$

(ii) In any given problem there are infinitely many choices of the pair u, v only a few or none of which may work.

EXAMPLE 8.4.2:

$$\int \underset{\underset{u}{\uparrow}}{\log x} \; \underset{\underset{dv}{\uparrow}}{dx} \qquad\qquad \text{Let} \quad u = \log x \quad v = x$$

$$= \underset{\underset{v}{\uparrow}}{x} \; \underset{\underset{u}{\uparrow}}{\log x} - \int \underset{\underset{v}{\uparrow}}{x} \cdot \underbrace{\frac{1}{x}}_{\underset{du}{\uparrow}} \, dx = x \log x - x + c.$$

EXAMPLE 8.4.3:

$$\int \underset{\underset{u}{\uparrow}}{x} \; \underbrace{e^x dx}_{\underset{dv}{\uparrow}} = \underset{\underset{u}{\uparrow}}{x} \; \underset{\underset{v}{\uparrow}}{e^x} - \int \underset{\underset{v}{\uparrow}}{e^x} \; \underset{\underset{du}{\uparrow}}{dx} = xe^x - e^x + c.$$

EXAMPLE 8.4.4:

$$\int \underset{\underset{u}{\uparrow}}{x} \; \underbrace{\sin x \; dx}_{\underset{dv}{\uparrow}} = \underset{\underset{u}{\uparrow}}{x} \; \underset{\underset{v}{\uparrow}}{(-\cos x)} - \int \underset{\underset{v}{\uparrow}}{(-\cos x)} \; \underset{\underset{du}{\uparrow}}{dx}$$

$$= -x \cos x + \sin x + c.$$

EXAMPLE 8.4.5:

$$\int \arctan x \; dx = x \arctan x - \int \frac{x}{1+x^2} \, dx = x \arctan x - \log \sqrt{1+x^2} + c$$

$$u = \arctan x \qquad\qquad dv = dx$$

$$du = \frac{1}{1+x^2} \, dx \qquad\qquad v = x.$$

EXAMPLE 8.4.6:

$$I = \int e^x \sin x \; dx = e^x \sin x - \int e^x \cos x \; dx$$

$$= e^x \sin x - \left(e^x \cos x + \int e^x \sin x \; dx \right)$$

$$= e^x (\sin x - \cos x) - I + c.$$

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Therefore $I = \frac{1}{2} e^x (\sin x - \cos x) + c$.

Alternatively:

$$I = \int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$= e^x (\sin x - \cos x) - I + c$$

so that $I = \frac{1}{2} e^x (\sin x - \cos x) + c$.

EXAMPLE 8.4.7:

$$I_n = \int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx = x^n e^x - n I_{n-1}.$$

The formula

$$I_n = x^n e^x - n I_{n-1}, \quad I_0 = e^x + c$$

is called a *Reduction Formula* and may be used to calculate I_n, for $n = 1, 2, \ldots$. For example to find $I_4 = \int x^4 e^x \, dx$, we find

$$I_4 = x^4 e^x - 4 I_3$$

$$= x^4 e^x - 4(x^3 e^x - 3 I_2)$$

$$= x^4 e^x - 4 x^3 e^x + 12(x^2 e^x - 2 I_1)$$

$$= x^4 e^x - 4 x^3 e^x + 12 x^2 e^x - 24(x e^x - I_0)$$

$$= x^4 e^x - 4 x^3 e^x + 12 x^2 e^x - 24 x e^x + 24 e^x + c.$$

EXAMPLE 8.4.8: Consider

$$I_n = \int \sin^n x \, dx = \int \sin^{n-1} x \, \sin x \, dx$$

$$= -\sin^{n-1} x \, \cos x + (n-1) \int \sin^{n-2} x \, \cos^2 x \, dx$$

$$= -\sin^{n-1} x \, \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) dx$$

$$= -\sin^{n-1} x \, \cos x + (n-1)I_{n-2} - (n-1)I_n \Rightarrow$$

$$nI_n = -\sin^{n-1} x \, \cos x + (n-1)I_{n-2} \Rightarrow$$

$$I_n = \frac{-1}{n} \sin^{n-1} x \, \cos x + \frac{n-1}{n} I_{n-2},$$

$$I_1 = -\cos x + c, \quad I_0 = x + c,$$

another example of a reduction formula. Note that we have another technique which is applicable to integrals of the form $\int \sin^n x \, dx$: if n is *odd* use the substitution $u = \cos x$ $\sin^2 x = 1 - u^2$; if n is *even* use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

Problems

8.5 Find $\int \log(x+3)dx$ by taking

    (a) $u = \log(x+3)$, $v = x$ [HINT: $\frac{x}{x+3} = 1 - \frac{3}{x+3}$],

    (b) $u = \log(x+3)$, $v = x+3$.

8.6 If $I_n = \int \cos^n x \, dx$, show $I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$,

$n = 1, 2, \ldots$.

8.7 Prove

    (a) $\int x \cos x \, dx = x \sin x + \cos x + c$,

    (b) $\int \text{Arcsin} \, dx = x \, \text{Arcsin} \, x + \sqrt{1-x^2} + c$,

    (c) $\int \text{Arctan} \, x \, dx = x \, \text{Arctan} \, x - \log \sqrt{1+x^2} + c$.

§8.5. Rational Functions.

A function r is rational if $r(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials. Here we consider antidifferentiation of such functions. Without less of generality it may be assumed that the degree of p is less than the degree of q; this can always be achieved by division. For example

$$\frac{x^4 + 9x^3 + 3}{x^2 + 1}$$

$$
\begin{array}{r}
x^2 + 9x \quad -1 \\
x^2 + 1 \overline{\smash{\big)}\ x^4 + 9x^3 + 3} \\
\underline{x^4 + x^2} \\
9x^3 - x^2 + 3 \\
\underline{9x^3 + 9x} \\
-x^2 - 9x + 3 \\
\underline{-x^2 - 1} \\
-9x + 4 \ .
\end{array}
$$

$$= x^2 + 9x - 1 + \frac{-9x + 4}{x^2 + 1} \ .$$

We develop the Method of Partial Fractions for such functions through several examples.

EXAMPLE 8.5.1: To evaluate $\int \frac{5x+4}{x^2-3x+2} \, dx$, we first factor the denominator so that $\frac{5x+4}{x^2-3x+2} = \frac{5x+4}{(x-1)(x-2)}$. We may write

$$\frac{5x + 4}{(x - 1)(x - 2)} = \frac{A}{x - 1} + \frac{B}{x - 2},$$

a partial fraction decomposition, if $5x + 4 = A(x - 2) + B(x - 1)$ for all x. One method of determining A, B is based on the fact that two polynomials in x are equal if and only if the coefficients of like powers of x are equal. Here we require

$$5 = A + B \quad \text{(coef. of } x^1)$$

$$4 = -2A - B \quad \text{(coef. of } x^0).$$

Thus $A = -9, B = 14$. Alternatively two polynomials of degree 1 such as we have here are equal for all x if and only if they agree at two distinct values of x. Two convenient values are $x = 1, x = 2$ which give directly $-9 = A, 14 = B$ as before. Now

$$\int \frac{5x+4}{x^2-3x+2} \, dx = \int \left(\frac{-9}{x-1} + \frac{14}{x-2} \right) dx$$
$$= \log |c(x-1)^{-9}(x-2)^{14}|.$$

Observe that any polynomial $p(x)$ of degree 1 or less may be expressed in the form $p(x) = A(x-2) + B(x-1)$ and so in this way we may evaluate $\int \frac{p(x)}{x^2-3x+2} \, dx$. The set of all such polynomials $p(x)$ is a vector space of dimension 2 and we have used the fact that the polynomials $x - 1, x - 2$ are a basis for this space.

EXAMPLE 8.5.2: Consider $\int \frac{2x^2-5x+3}{(x-1)^2(x-2)} \, dx$. Here we may write

$$\frac{2x^2 - 3x + 3}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

provided $2x^2 - 3x + 3 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$. Therefore

$$\left.
\begin{aligned}
(x = 1) &\Rightarrow 2 = -B \\
(x = 2) &\Rightarrow 5 = C \\
(\text{coef. } x^2) &\Rightarrow 2 = A + C
\end{aligned}
\right\} \Rightarrow A = -3, B = -2, C + 5$$

so that $\int \frac{2x^2-5x+3}{(x-1)^2(x-2)} \, dx = \log \left| \frac{(x-2)^5}{(x-1)^3} \right| + \frac{2}{x-1} + c$. We have used the fact that the functions $(x-1)(x-2), (x-2), (x-1)^2$ form a basis for the vector space of polynomials of degrees 3 or less. Also instead of equating the coefficients of x^2 we could have equated

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the coefficients of x^1 or x^0 or considered one more value of x besides $x = 1, 2$.

EXAMPLE 8.5.3:

$$\frac{x^3 + 3}{x^2 + x} = x - 1 + \frac{x + 3}{x^2 + x}$$

$$\begin{array}{r} x-1 \\ x^2 + x \overline{\smash{\big)}\, x^3 + 3} \\ \underline{x^3 + x^2} \\ -x^2 + 3 \\ \underline{-x^2 - x} \\ x + 3 \end{array}$$

$$= x - 1 + \frac{A}{x} + \frac{B}{x + 1}$$

if $x + 3 = A(x + 1) + Bx$.

$$(x = -1) \Rightarrow 2 = -B$$

$$(x = 0) \Rightarrow 3 = A$$

$$\int \frac{x^3 + 3}{x^2 + x} \, dx = \int (x - 1 + \frac{3}{x} - \frac{2}{x + 1}) dx$$

$$= \frac{1}{2} x^2 - x + \log \left| \frac{x^3}{(x + 1)^2} \right| + c.$$

EXAMPLE 8.5.4:

$$\frac{1}{(x - 1)(x^2 + 2x + 2)} = \frac{A}{(x - 1)} + \frac{Bx + C}{x^2 + 2x + 2}$$

$$\Leftrightarrow 1 = A(x^2 + 2x + 2) + (Bx + C)(x - 1), \quad \text{for all} \quad x$$

Therefore

$$\left. \begin{array}{l} (x = 1) \Rightarrow 1 = 5A \\ (\text{coef. } x^2) \Rightarrow 0 = A + B \\ (\text{coef. } x^0) \Rightarrow 1 = 2A - C \end{array} \right\} \Rightarrow A = \frac{1}{5}, \quad B = -\frac{1}{5}, \quad C = -\frac{3}{5}$$

$$\int \frac{1}{(x-1)(x^4+2x+2)}\,dx = \frac{1}{5}\int(\frac{1}{x-1} - \frac{x+3}{x^2+2x+2})\,dx$$

$$= \frac{1}{5}\int(\frac{1}{x-1} - \frac{1}{2}\frac{2x+2}{x^2+2x+2} - \frac{2}{x^2+2x+2})\,dx$$

$$= \frac{1}{5}\log|x-1| - \frac{1}{10}\log(x^2+2x+2)$$

$$- \frac{2}{5}\arctan(x+1) + c.$$

In this example we have used the fact that the polynomials $x^2+2x+2, x(x-1), x-1$ are a basis for the three dimensional vector space of polynomials of degree 2 or less. We would not have a basis if we omitted any of the A, B, C terms. Notice also that we have written

$$\int \frac{x+3}{x^2+2x+2}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx + 2\int \frac{1}{x^2+2x+2}\,dx.$$

The first integral on the right is of the form $\frac{1}{2}\int \frac{du}{u} = \log|u| + c$ and has been chosen to take care of the 'x term' in the numerator. The other integral on the right is the difference of the previous two and is of the form $2\int \frac{dv}{a^2+v^2} = \frac{2}{a}\arctan(\frac{v}{a}) + k$, where $v = x+1, a = 1$.

EXAMPLE 8.5.5:

$$\int \sec\theta\,d\theta = \log|\sec\theta + \tan\theta| + c.$$

We have given the verification of this formula as Problem 8.3; this is easily done by differntation of the right-hand side. However, 'discovery' of the formula is a little more subtle as we saw in Example 8.3.7.

Now we can make more progress

$$\int \sec \theta \, d\theta = \int \frac{1}{\cos \theta} d\theta = \int \frac{\cos \theta}{\cos^2 \theta} \, d\theta = \int \frac{\cos \theta}{1 - \sin^2 \theta} \, d\theta$$

$$= \int \frac{du}{1 - u^2} = \frac{1}{2} \int (\frac{1}{1 - u} + \frac{1}{1 + u}) du \qquad \begin{array}{l} u = \sin \theta, \; du = \cos \theta \, d\theta \\ 1 - u^2 = 1 - \sin^2 \theta \end{array}$$

(partial fractions)

$$= \log \sqrt{\frac{1 + u}{1 - u}} + c = \log \sqrt{\frac{1 + \sin \theta}{1 - \sin \theta}} + c = \log \sqrt{\frac{(1 + \sin \theta)^2}{\cos \theta^2}} + c$$

$$= \log |\sec \theta + \tan \theta| + c.$$

We can now describe the Method of Partial Fractions

STEP 1. If $r(x) = \frac{p(x)}{q(x)}$ is rational we begin by writing it in the form

$$r(x) = t(x) + \frac{s(x)}{q(x)}, \quad t, s, q \quad \text{polynomials and}$$

where s is a polynomial whose degree is less than that of the denominator q. This may be achieved by division.

STEP 2. Factor the denominator q in the form

$$q(x) = (x - a)^\ell \ldots (x^2 + \alpha x + \beta)^m \ldots$$

and expand $\frac{s}{q}$ in terms of partial fractions

$$\frac{s(x)}{q(x)} = (\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2}$$

$$+ \cdots + \frac{A_\ell}{(x-a)^\ell}) + \cdots + (\frac{P_1 x + Q_1}{x^2 + \alpha x + \beta} + \cdots + \frac{P_m x + Q_m}{(x^2 + \alpha x + \beta)^m}).$$

Now $\int r(x)dx = \int t(x)dx + \int \frac{s(x)}{q(x)} dx$. Antidifferentiation of $t(x)$ is easy; so also are the terms $\int \frac{A_k}{(x-a)^k} dx = A_1 \log|x - a| + c$, if $k = 1$, $\frac{A_k}{(1-k)} (x-a)^{1-k} + c$, if $k > 1$. Finally $\int \frac{P_k x + Q_k}{(x^2 + \alpha x + \beta)^k} dx$ is written in the form $\frac{1}{2} P_k \int \frac{du}{u^k} + R_k \int \frac{dv}{(\lambda^2 + v^2)^k}$, $u = x^2 + \alpha x + \beta, v = x + \frac{\alpha}{2}$.

The first of these integrals is a logarithm $(k = 1)$ or a power of u $(k > 1)$. In the second, use

$$\int \frac{dv}{(\lambda^2 + v^2)^k} = \int \frac{\lambda \sec^2 \theta \, d\theta}{\lambda^{2k} \sec^{2k} \theta} \qquad v = \lambda \tan \theta, \quad dv = \lambda \sec^2 \theta \, d\theta$$
$$\lambda^2 + v^2 = \lambda^2 \sec^2 \theta$$

$$= \lambda^{1-2k} \int \cos^{2k-2} \theta \, d\theta.$$

This may be handled by a method of §8.3 $(m, n$ both even). However, the reduction formula of Problem 8.6 is very useful here.

Problems

8.8 Show

(a) $\int \frac{1}{x^2-x} \, dx = \log \left| \frac{x-1}{cx} \right|$,

(b) $\int \frac{3x+2}{x^2-x-2} \, dx = \frac{1}{3} \log |c(x+1)(x-2)^8|$,

(c) $\int \frac{5x^2+3x+2}{x^2-x-2} \, dx = 5x + \frac{4}{3} \log \left| \frac{(x-2)^7}{x+1} \right| + c$,

(d) $\int \frac{5x^2+4x-2}{(x+1)(x-1)(x-2)} \, dx = \log \left| c \, \frac{(x-2)^{\frac{26}{3}}}{(x+1)^{\frac{1}{6}}} (x-1)^{\frac{1}{2}} \right|$.

8.9 Show

(a) $\int \frac{x^2-x+1}{(x-1)^2(x-2)} \, dx = \frac{1}{x-1} + \log \left| \frac{c(x-2)^3}{(x-1)^2} \right|$,

(b) $\int \frac{1}{(x^2+2x+2)(x-3)} \, dx = \frac{1}{17} [\log |x-3|$
$\qquad - \frac{1}{2} \log |x^2 + 2x + 2| - 4 \operatorname{Arctan}(x+1)] + c$,

(c) $\int \frac{3x+2}{(x^2+2x+5)} \, dx = \frac{3}{2} \log(x^2+2x+5) - \frac{1}{2} \operatorname{Arctan}(\frac{x+1}{2}) + c$,

(d) $\int \frac{1}{x^4+4} \, dx = \frac{1}{8} [\log \sqrt{\frac{x^2+2x+2}{x^2-2x+2}}$
$\qquad + \operatorname{Arctan}(x+1) + \operatorname{Arctan}(x-1)] + c$.

§8.6. Another Substitution.

The half-angle substitution is

$$t = \tan \frac{1}{2} \theta.$$

Then

$$\cos \theta = \frac{1-t^2}{1+t^2}, \quad \sin \theta = \frac{2t}{1+t^2}, \quad d\theta = \frac{2dt}{1+t^2},$$

so that

$$\int f(\cos \theta, \sin \theta)d\theta = \int f\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \frac{2dt}{1+t^2}.$$

Thus this substitution transforms the antidifferentation of a rational function of sines and cosines to the antidiffernetation of a rational function of t – the subject of the last section. A word of caution however: this is more or less a technique of last resort and you should check that there may be a simpler way of tackling the question.

To establish these statements observe that, if $t = \tan(\frac{\theta}{2})$,

$$1 + \cos \theta = 2\cos^2\left(\frac{\theta}{2}\right) = \frac{2}{\sec^2(\frac{\theta}{2})} = \frac{2}{1+\tan^2(\frac{\theta}{2})} = \frac{2}{1+t^2}$$

and so $\cos \theta = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$. Next

$$\sin \theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = 2\tan\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)$$

$$= \frac{2\tan(\frac{\theta}{2})}{\sec^2(\frac{\theta}{2})} = \frac{2t}{1+t^2},$$

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and

$$dt = \frac{1}{2} \sec^2(\frac{\theta}{2})d\theta = \frac{1}{2}[1 + \tan^2(\frac{\theta}{2})]d\theta, \quad \text{which implies}$$

$$d\theta = \frac{2dt}{1 + t^2}.$$

EXAMPLE 8.6.1:

$\int \frac{dx}{3 + \cos x}$ $\qquad\qquad\qquad t = \tan \frac{1}{2} x$

$\qquad = \int \frac{1}{3 + \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$ $\qquad\qquad \frac{2dt}{1+t^2} = dx, \ \frac{1-t^2}{1+t^2} = \cos x$

$\qquad = \int \frac{2dt}{4+2t^2} = \int \frac{dt}{2+t^2}$

$\qquad = \frac{1}{\sqrt{2}} \arctan(\frac{t}{\sqrt{2}}) + c$

$\qquad = \frac{1}{\sqrt{2}} \arctan(\frac{1}{\sqrt{2}} \tan \frac{1}{2} x) + c.$

EXAMPLE 8.6.2:

$$\int \frac{dx}{1 + \cos x} = \int \frac{\frac{2dt}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}} = \int dt = t + c = \tan(\frac{1}{2} x) + c.$$

Alternatively

$$\int \frac{dx}{1 + \cos x} = \int \frac{dx}{2\cos^2(\frac{x}{2})} = \frac{1}{2} \int \sec^2(\frac{x}{2})dx = \tan(\frac{1}{2} x) + c.$$

EXAMPLE 8.6.3:

$$\int \frac{\sin x}{3 + \cos x} \, dx = -\log[3 + \cos x] + c,$$

by the substitution $u = 3 + \cos x$. This is quite straight forward.
On the other hand, the method of this section makes this a fairly

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tedious exercise.

$$\int \frac{\sin x}{3 + \cos x}\, dx = \int \frac{\frac{2t}{1+t^2}}{3 + \frac{(1-t^2)}{(1+t^2)}} \cdot \frac{2\,dt}{1+t^2}$$

$$t = \tan \tfrac{1}{2} x$$

$$\frac{1-t^2}{1+t^2} = \cos x$$

$$= 4 \int \frac{t}{(1+t^2)(4+2t^2)}\, dt$$

$$\frac{2t}{1+t^2} = \sin x$$

$$\frac{2\,dt}{1+t^2} = dx$$

$$= \int \frac{du}{(1+u)(2+u)}$$

$$u = t^2,\; du = 2t\, dt$$

$$= \int \left(\frac{1}{1+u} - \frac{1}{2+u}\right) du$$

(method of partial fractions)

$$= \log\left| \frac{1+u}{2+u} \right| + c$$

$$= \log\left[\frac{1+t^2}{2+t^2} \right] + c$$

$$= \log\left[\frac{1+\tan^2 \tfrac{1}{2} x}{2+\tan^2 \tfrac{1}{2} x} \right] + c$$

$$= \log\left[\frac{\sec^2 \tfrac{1}{2} x}{1+\sec^2 \tfrac{1}{2} x} \right] + c$$

$$= -\log\left[\cos^2 \tfrac{1}{2} x + 1 \right] + c$$

$$= -\log\left[\tfrac{1}{2}(1 + \cos x) + 1 \right] + c$$

$$= -\log\left[3 + \cos x \right] + k.$$

Problems

8.10. Evaluate

(a) $\int_0^\pi \sin(\frac{x}{2})dx$,   (b) $\int_0^1 \frac{x}{x^2-4} dx$,   (c) $\int_0^1 \frac{1}{x^2-4} dx$,

(d) $\int_1^e \frac{\sqrt{\log x}}{x} dx$,   (e) $\int_0^{\frac{\pi}{2}} \sin x \cos^2 x \, dx$,   (f) $\int_0^{\frac{\pi}{4}} \sin 3x \cos x \, dx$
$$[\sin a \cos b =?]$$

8.11. Find the antiderivatives

(a) $\int 2^x dx$,   (b) $\int \frac{x}{\sqrt{x+4}} dx$,   (c) $\int \sin^2 x dx$,

(d) $\int \sqrt{4-x^2}\, dx$,   (e) $\int \frac{2x+3}{x(x^2+16)} dx$,   (f) $\int \frac{e^{2x}-1}{e^{2x}+1} dx$.

8.12. Evaluate

(a) $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \, dx$,   (b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1+\tan^3 x) \sec^2 x \, dx$,   (c) $\int_0^{\frac{\pi}{2}} (x+1) \cos x \, dx$,

(d) $\int_a^b \cos(3x-1) dx$,   (e) $\int_0^{\frac{\pi}{4}} \sec^4 x \, dx$,   (f) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x} dx$.

8.13. Evaluate

(a) $\int_1^2 \frac{x-2}{(x+1)x} dx$,   (b) $\int_0^1 \frac{x-2}{(x+1)^2} dx$,   (c) $\int_0^2 \frac{1}{4+x^2} dx$,

(d) $\int_0^2 \frac{1}{(4+x^2)^2} dx$,   (e) $\int_0^1 \frac{x^2}{6+x^3} dx$,   (f) $\int_0^1 \frac{x\, dx}{x^2+x+1}$.

8.14. Use the substitution $t = \tan \frac{1}{2}\theta$ to evaluate

$$\text{(a)} \int \frac{d\theta}{1+3\sin\theta}, \qquad \text{(b)} \int \sec\theta\, d\theta.$$

Reconcile your answer in (b) with that in Example 8.5.5.

8.15. Find   (a) $\int \frac{1}{x^3+1} dx$,   (b) $\int \frac{1}{x^3-1} dx$.

HINT: You should only have to do one of these.

8.16. Find   (a) $\int \frac{1-\tan x}{1+\tan x} dx$,   (b) $\int e^{e^x+x} dx$.

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8.17. Show $\int_0^1 \frac{dt}{1+t^3} = \frac{\pi + \sqrt{3} \log 2}{3\sqrt{3}}$.

8.18. Show $\int_0^{\frac{\pi}{2}} \sin^2 2\theta \cos \theta \, d\theta = \frac{8}{15}$.

8.19. Find a reduction formula for $\int (\log x)^n dx$ and use the formula to compute $\int_1^e (\log x)^5 dx$.

8.20. Find $\int \frac{du}{u^4 + 1}$.

8.21. Find

     (a) $\int_0^{\frac{\pi}{2}} x \sin x \cos x \, dx$,   (b) $\int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}} \, dx$,

     (c) $\int_0^t \frac{x}{\sqrt{1-x^2}} \, dx$,   (d) $\int_0^t \frac{x^2}{\sqrt{1-x}} \, dx$.

8.22. Show $\int_0^{\frac{\pi}{4}} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{1}{ab} \operatorname{Arctan}(\frac{a}{b})$, if $a, b > 0$.

8.23. Find a reduction formula for $\int \frac{1}{(a^2+u^2)^n} \, du$.

8.24. Show $\int \csc \theta \, d\theta = -\log |\csc \theta + \cot \theta| + c$.

8.25. Show $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log |x + \sqrt{x^2 \pm a^2}| + c$.

8.26. Show $\int_a^b (x-a)^m (b-x)^n dx = \frac{m! n!}{(m+n+1)!} (b-a)^{m+n+1}$.

8.27. Let $I = \int e^{ax} \cos bx \, dx$, $J = \int e^{ax} \sin bx \, dx$. Find I, J.

    HINT: Integrate each by parts once. Solve the resulting equations for I, J.

8.28. Find   (a) $\int \log(1 + x^2) dx$,   (b) $\int \arcsin(\frac{1}{x}) \, dx$.

8.29. Show

    (a) $\int_{-\pi}^{\pi} \cos mx \, \sin nx \, dx = 0,$ $m, n = 0, 1, 2, \ldots$

    (b) $\int_{-\pi}^{\pi} \cos mx \, \cos nx \, dx = 0$ and

    (c) $\int_{-\pi}^{\pi} \sin mx \, \sin nx \, dx = 0,$ if $m, n = 0, 1, 2, \ldots, \; m \neq n$

    (d) $\int_{-\pi}^{\pi} \cos^2 mx \, dx \; = \begin{cases} 2\pi, & m = 0 \\ \pi, & m = 1, 2, \ldots \end{cases}$

    (e) $\int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi,$ $m = 1, 2, \ldots .$

8.30. A function f is *even* if $f(-x) = f(x)$ for all x. For example $x^2, \cos x$ are even. An *odd* function is one for which $f(-x) = -f(x)$ for all x. The functions $x, \sin x$ are odd. Suppose $\int_{-a}^{a} f$ exists. Show

    (a) $\int_{-a}^{a} f = 2 \int_{0}^{a} f$ if f is even.

    (b) $\int_{-a}^{a} f = 0$ if f is odd.

IX. APPLICATIONS OF INTEGRATION

§9.1 Areas and Volumes.

We define area and volume for certain sets subject to reasonable requirements on the sets involved and on the concepts of area and volume. We will consider area for sets S in the plane which lie in some strip $\{(x,y) : a \leq x \leq b\}$.

We set
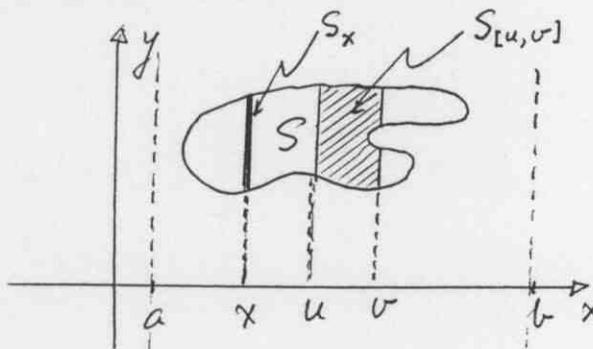
$$S_x = \{y : (x,y) \in S\}, \quad \text{if} \quad a \leq x \leq b$$

$$S_{[u,v]} = \{(x,y) \in S : u \leq x \leq v\}, \quad \text{if} \quad [u,v]$$

is an interval, and impose the following requirements:

(i) If $u < v < w$, then

$$\text{area } S_{[u,w]} = \text{area } S_{[u,v]} + \text{area } S_{[v,w]}.$$

(ii) If $a \leq x \leq b$, then

$$\text{length } S_x = \ell(x).$$

(iii) If $m \leq \ell(x) \leq M$ when $u \leq x \leq v$, then

$$m(v - u) \leq \text{area } S_{[u,v]} \leq M(v - u).$$

We then write

$$\text{area } S = \int_a^b \ell(x)dx,$$

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if the integral exists.

The motivation for this is that, if $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, then

$$\text{area } S = \sum_{n=1}^{n} \text{area } S_{[x_{k-1}, x_k]}$$

from (i). From (iii), we find that if $\overset{M_n}{M_k} \leq \ell(x) \leq M_k$, when $x_{k-1} \leq x \leq x_k$, then

$$m_k(x_k - x_{k-1}) \leq \text{area } S_{[x_{k-1}, x_k]} \leq M_k(x_k - x_{k-1}).$$

Therefore

$$L(P, \ell) \leq \text{area } S \leq U(P, \ell)$$

and since $\int_a^b \ell$ is the unique number satisfying $L(P, \ell) \leq \int_a^b \ell \leq U(P, \ell)$ we define area $S = \int_a^b \ell$.

PROPOSITION 9.1.1.. *Suppose f and g are continuous on $[a, b]$ and $f(x) \leq g(x), a \leq x \leq b$. Then the set $S = \{(x, y) : a \leq x \leq b, f(x) \leq y \leq g(x)\}$ has area*

$$A = \int_a^b [g(x) - f(x)]dx.$$

In this case

$\ell(x) = g(x) - f(x)$,

ℓ is continuous on $[a, b]$ so that

$\int_a^b \ell$ exists and the area is

given by the formula as stated.

Another way to approach this problem is to suppose ℓ is continuous on $[a, b]$ and let $A(x) = $ area $S_{[a,x]}$ and

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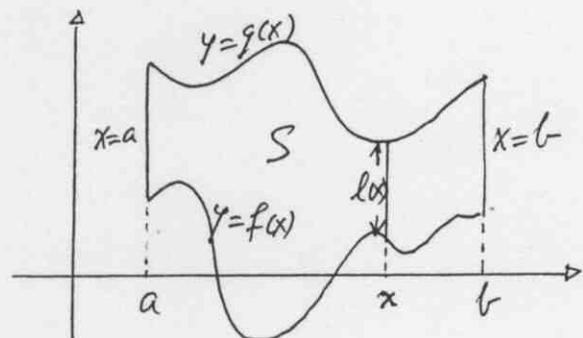
$$\triangle A = A(x + \triangle x) - A(x),$$

$$= \begin{cases} S_{[x, x+\triangle x]}, & \text{if} \quad \triangle x > 0, \\ \\ - \text{ area } S_{[x+\triangle x, x]}, & \text{if} \quad \triangle x < 0. \end{cases}$$

From (iii)
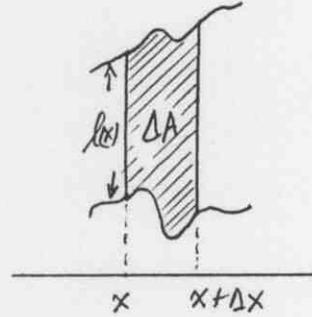
$\ell(x_*)\triangle x \le \triangle A \le \ell(x^*)\triangle x$ where $\ell(x_*), \ell(x^*)$ are the minimum

and maximum respectively of ℓ on the interval $[a, x + \triangle x]$, if

$\triangle x > 0$. A similar inequality is satisfied if $\triangle x < 0$. Since

$$\lim_{\triangle x \to 0} \ell(x_*) = \ell(x), \quad \lim_{\triangle x \to 0} \ell(x^*) = \ell(x),$$

we have by the Squeeze Principle

$$\lim_{\triangle x \to 0} \frac{\triangle A}{\triangle x} = \ell(x)$$

so that $A'(x) = \ell(x)$, $a \le x \le b$, and

$$A = A(b) - A(a) = \int_a^b A'(x)dx = \int_a^b \ell(x)dx.$$

This argument is summarized formally as

$$dA = \ell(x)dx$$

$$A = \int_{x=a}^{x=b} dA = \int_a^b \ell(x)dx.$$
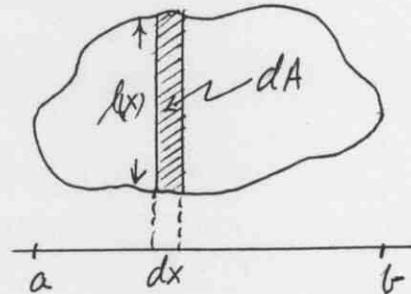
EXAMPLE 9.1.2: The area of the region bounded by the curves

$x = -1, y = x, y = x^2$ is

$\int_{-1}^1 |x^2 - x|dx$

$= \int_{-1}^0 (x^2 - x)dx + \int_0^1 (x - x^2)dx$

$= (\frac{1}{3}x^3 - \frac{1}{2}x^2)|_{-1}^0 + (\frac{1}{2}x^2 - \frac{1}{3}x^3)|_0^1$

$= (\frac{1}{3} + \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) = \frac{5}{6} + \frac{1}{6} = 1.$

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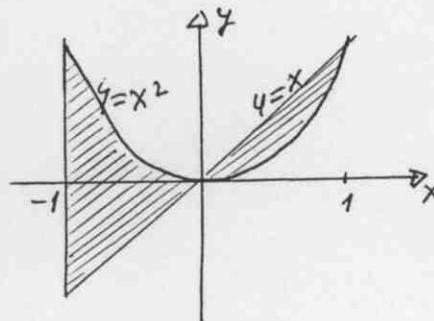
EXAMPLE 9.1.3: The area of the circle

$x^2 + y^2 \leq a^2$ is

$$2 \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = 2a^2 \int_{-\pi/2}^{\pi/2} \cos^2 \theta\, d\theta$$

$$= a^2 \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta$$

$$= a^2 (\theta + \tfrac{1}{2} \sin 2\theta)|_{-\pi/2}^{\pi/2}$$

$$= \pi a^2.$$

$x = a \sin \theta \quad dx = a \cos \theta\, d\theta$

$\sqrt{a^2 - x^2} = a \cos \theta$

$-a \leq x \leq a \longleftrightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

EXAMPLE 9.1.4: As motivation for a later example (9.1.8) we find the formula for the area of a triangle. Choose the origin O at the vertex with the y-axis parallel to the base. Here $\ell(x)$ is proportional to x : $\ell(x) = kx$. Hence

$$\text{Area} = \int_0^h kx\, dx = \frac{1}{2}\, kx^2|_0^h = \frac{1}{2}\, kh^2 = \frac{1}{2}\, bh$$

since $b = \ell(h) = kh$.

We now consider the volume of solids by the *Method of Sections*.
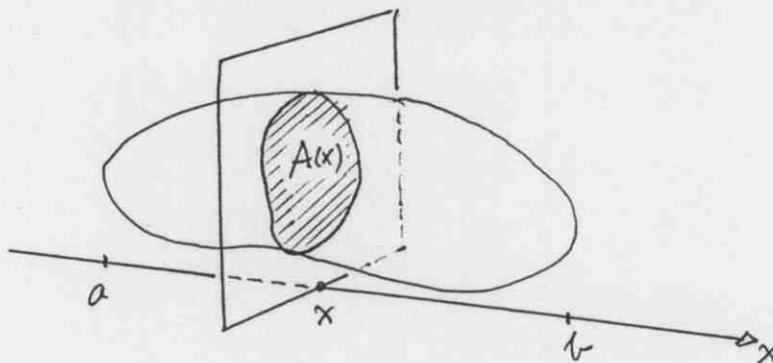
Suppose the solid S lies between two planes perpendicular to

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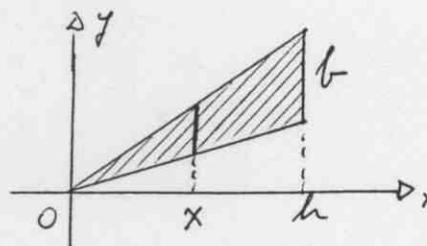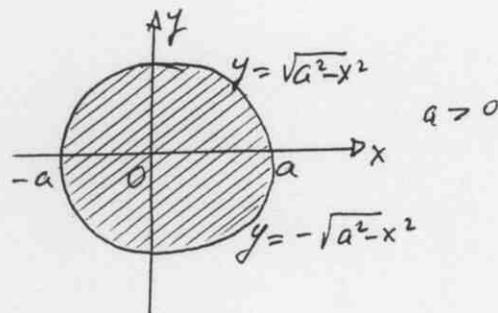
a certain axis and through the points $a, b \, (a < b)$ on that axis. We denote by S_x the intersection of S with the plane perpendicular to the axis and through the point x, and by $S_{[u,v]}$ the portion of S which lies between the perpendicular planes through u, v. We will impose the requirements:

(i) If $u < v < w$, then

$$\text{volume } S_{[u,w]} = \text{volume } S_{[u,v]} + \text{volume } S_{[v,w]}.$$

(ii) If $a \leq x \leq b$, then

$$\text{Area } S_x = A(x).$$

(iii) If $m \leq A(x) \leq M$ when $u \leq x \leq v$, then

$$m(v - u) \leq \text{volume } S_{[u,v]} \leq M(v - u).$$

We then write

$$\text{volume } S = \int_a^b A(x)dx,$$

if the integral exists.

The motivation for this, as in the case of areas, is that, if $P = \{x_0, \ldots, x_n\}$ is a partition of $[a, b]$, then

$$\text{volume } S = \sum_{k=1}^{n} \text{volume } S_{[x_{k-1}, x_k]}.$$

Also, if $m_k \leq A(x) \leq M_k$ when $x_{k-1} \leq x \leq x_k$, then

$$m_k(x_k - x_{k-1}) \leq \text{volume } S_{[x_{k-1}, x_k]} \leq M_k(x_k - x_{k-1})$$

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which implies

$$L(P, A) \leq \text{ volume } S \leq U(P, A)$$

for all partitions P of $[a, b]$, giving volume $S = \int_a^b A$.

Another approach is to suppose A is continuous on $[a, b]$ and let

$$V(x) = \text{ volume } S_{[0, x]}$$

$$\Delta V = V(x + \Delta x) - V(x)$$
$$= \begin{cases} \text{volume } V_{[x, x+\Delta x]}, & \text{if } \Delta x > 0 \\ \\ - \text{ volume } V_{[x+\Delta x, x]}, & \text{if } \Delta x < 0. \end{cases}$$

As in the case of areas, there exist x_*, x^* between x and $x + \Delta x$ such that, from (iii),

$$A(x_*)\Delta x \leq \Delta V \leq A(x^*)\Delta x$$

which implies $\lim\limits_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = A(x)$, or $V'(x) = A(x)$. Thus $V = V(b) - V(a) = \int_a^b V'(x)dx = \int_a^b A(x)dx$.

This argument is formally summarized as

$$dV = A(x)dx$$
$$V = \int_{x=a}^{x=b} dV = \int_a^b A(x)dx.$$
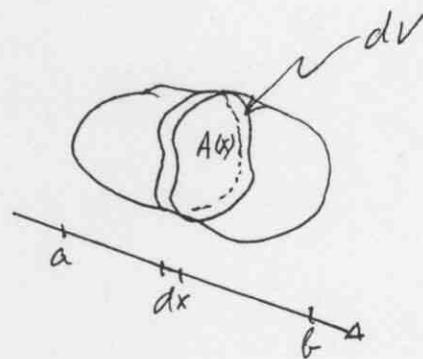
PROPOSITION 9.1.5. (SOLIDS OF REVOLUTION).

Let f be continuous on $[a, b]$. Suppose the solid
S is generated by revolving the area bounded by the curves
$y = f(x), y = 0, x = a, x = b$

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about the x-axis.

Then, since $A(x) = \pi f(x)^2$,

we have the volume

$$V = \pi \int_a^b f(x)^2 dx.$$
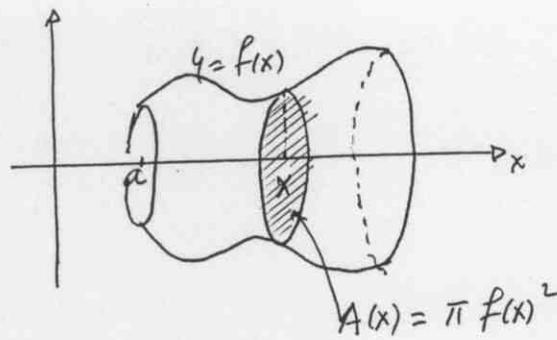
EXAMPLE 9.1.6: The volume of a sphere of radius a is given by

$$\begin{aligned}
V &= \pi \int_{-a}^a (a^2 - x^2) dx \\
&= \pi(a^2 x - \frac{x^3}{3})|_{-a}^a \\
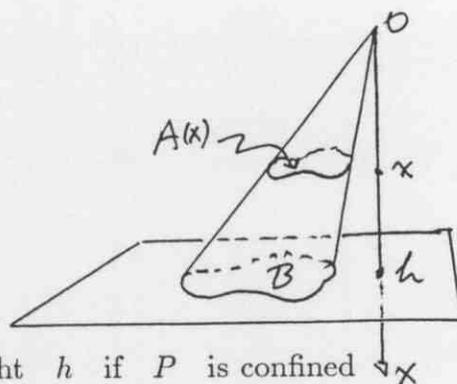&= \frac{4}{3} \pi a^3.
\end{aligned}$$
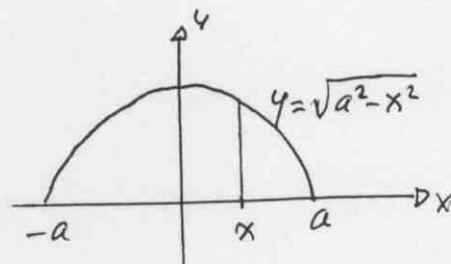
EXAMPLE 9.1.7: The volume of a right

circular cone of base

radius a and height h is

$$\begin{aligned}
V &= \pi \int_0^h (\frac{a}{h} x)^2 dx \\
&= \pi \frac{a^2}{h^2} \frac{x^3}{3} |_0^h \\
&= \frac{1}{3} \pi \frac{a^2}{h^2} h^3 \\
&= \frac{1}{3} \pi a^2 h.
\end{aligned}$$
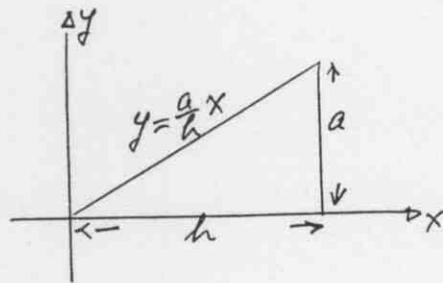
EXAMPLE 9.1.8: The preceding example is a special case of the for-
mula for the volume of a general cone.

Let B be the area of a region in the plane and O a point at a distance h from the plane. The lines OP generate a cone of base area B and height h if P is confined to the region in the plane. If $A(x)$ is the area of the planar section at a depth x below the vertex, then $A(x) = kx^2$ where k is constant. It is easy to see this for rectangular regions and so it follows for general regions which may be tiled with rectangles. Then

$$V = \int_0^h A(x)dx = \int_0^h kx^2 dx = \frac{1}{3} kh^3 = \frac{1}{3} Bh$$

since $B = A(h) = kh^2$.

The Method of Sections may not be always the most convenient method for computing volumes. For example, if we generate a solid by rotating the region $\{(x, y) : a \leq x \leq b, \ 0 \leq y \leq f(x)\}$, $a > 0$, about the y-axis, then the areas of sections perpendicular to this axis may be quite complicated. It may be more convenient to use the *Method of Cylindrical Shells*. Suppose f is continuous on $[a, b]$ and let $V(x)$ be the volume of that portion of the solid which is a distance x or less from the
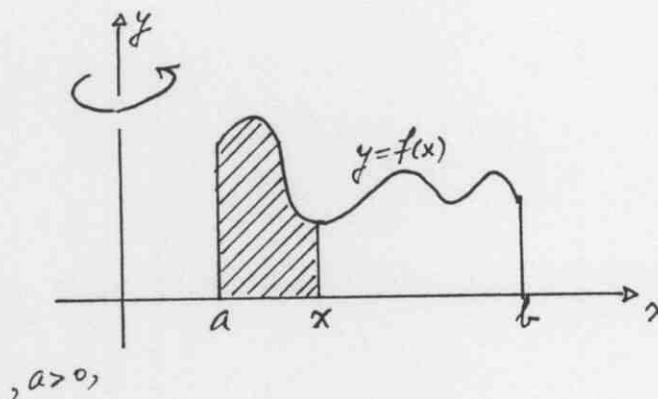
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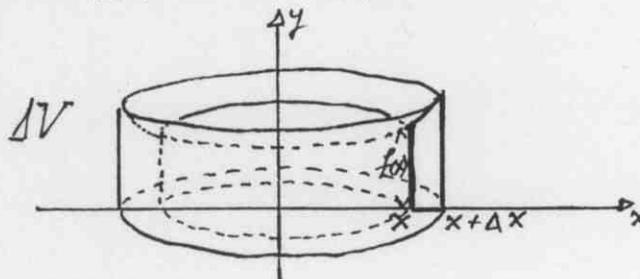
y axis. Let

$$\Delta V = V(x + \Delta x) - V(x);$$

$$\pi[(x + \Delta x)^2 - x^2]f(x_*) \le \Delta V \le \pi[(x + \Delta x)^2 - x^2]f(x^*)$$

where $f(x_*), f(x^*)$ are extrema of f between x and $x + \Delta x$.
Dividing by Δx,   we find

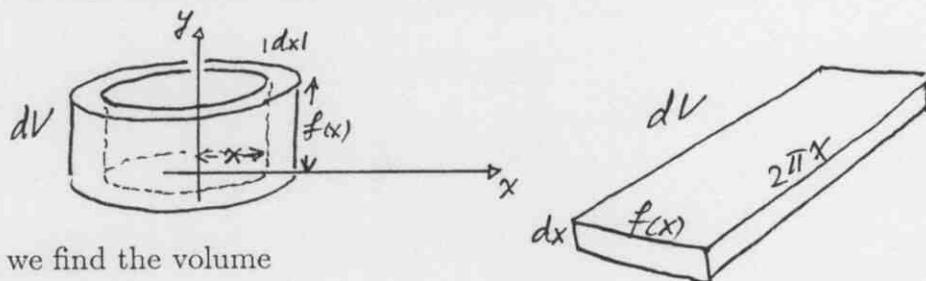$$\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = 2\pi x f(x)$$

so that $V'(x) = 2\pi x f(x)$ and, if $V = V(b) - V(a)$,

$$V = 2\pi \int_a^b x f(x) dx.$$

A formal summary of the Method of Shells is as follows:

$dV = 2\pi x f(x) dx$

$V = \int_{x=a}^{x=b} dV$

$\quad = 2\pi \int_a^b x f(x) dx.$

EXAMPLE 9.1.9: Again we find the volume

of a right circular cone,

this time by the

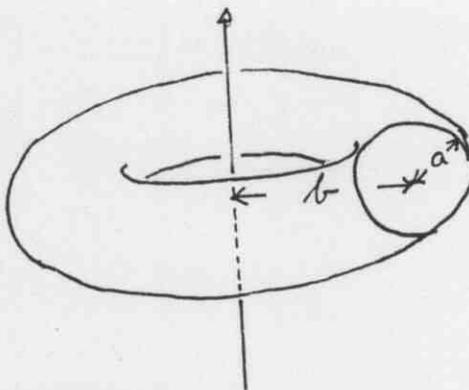Method of Shells.

$$V = 2\pi \int_0^a x(-\frac{h}{a} x + h) dx$$

$$= 2\pi[-\frac{h}{a} \frac{1}{3} a^3 + h \frac{1}{2} a^2]$$

$$= \frac{1}{3} \pi a^2 h.$$
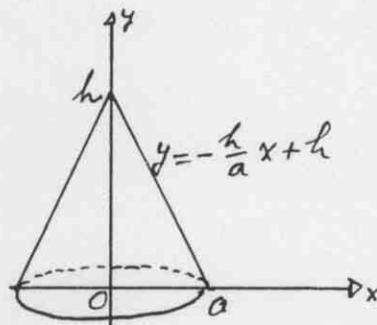
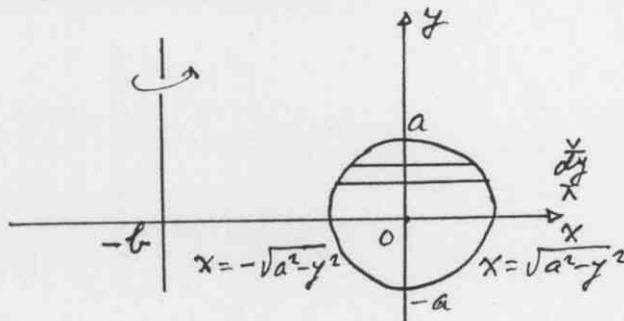EXAMPLE 9.1.10: A torus is generated

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by rotating a disc of
radius a about
an axis in the plane
of the disc at a distance
$b > a$ from the center
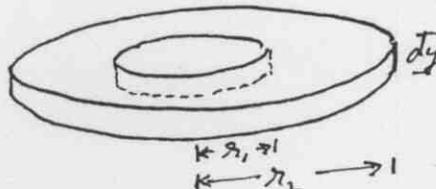of the disc. The volume V of the torus is given by

$$V = 2\pi^2 a^2 b.$$

We prove this by the Method of Sections and by the Method of Shells.

It is convenient here
to generate the torus
by rotating the disc
$x^2 + y^2 \leq a^2$ about the
line $x = -b$.

Method 1 (Sections)

$$dV = [\pi(b + \sqrt{a^2 - y^2})^2 - \pi(b - \sqrt{a^2 - y^2})^2]dy$$

$$= \pi[b^2 + 2b\sqrt{a^2 - y^2} + (a^2 - y^2) - b^2 + 2b\sqrt{a^2 - y^2} - (a^2 - y^2)]dy \qquad r_1 = b - \sqrt{a^2 - y^2}$$

$$\qquad\qquad r_2 = b + \sqrt{a^2 - y^2}$$
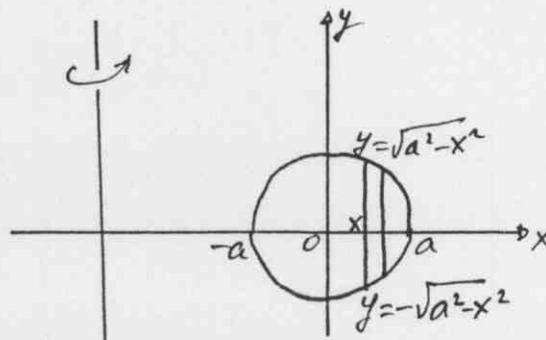
$$= \pi 4b\sqrt{a^2 - y^2}\ .$$

Thus $V = 4\pi b \int_{-a}^{a} \sqrt{a^2 - y^2}\ dy = 4\pi b \cdot \frac{\pi}{2} a^2 = 2\pi^2 a^2 b.$

Method 2 (Shells)

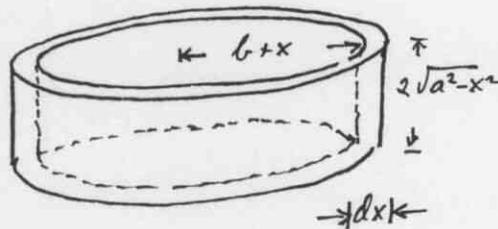$$dV = 2\pi(b + x)2\sqrt{a^2 - x^2}\ dx$$

$$= 4\pi(b + x)\sqrt{a^2 - x^2}\ dx.$$

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$$V = 4\pi \int_{-a}^{a} (b+x)\sqrt{a^2 - x^2}\, dx$$

$$= 4\pi b \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx, \quad \text{since} \quad \int_{-a}^{a} x\sqrt{a^2 - x^2}\, dx = 0 \quad \text{(Why?)}$$

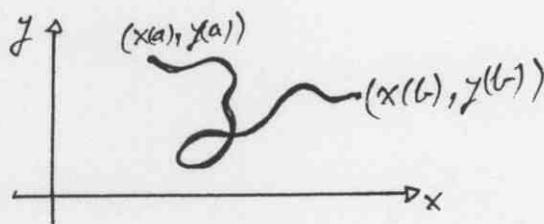$$= 4\pi b \cdot \frac{\pi}{2}\, a^2 = 2\pi^2 a^2 b.$$

§9.2. Arc Length and Surface Area.

If $x(t), y(t)$ are differentiable functions $a \leq t \leq b$, then the equations

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b$$

describe a smooth curve in $\mathbb{R}^2$. Here t is called a parameter and we call this a parametric representation of the curve. You can think of t as time and $(x(t), y(t))$ as the coordinates of a moving point at time t, $a \leq t \leq b$. Our basic 'reasonable requirement' on arc length here is that the ratio
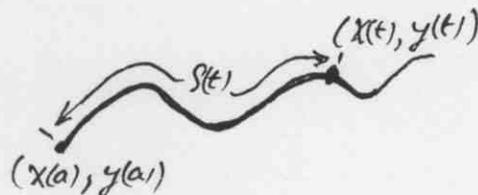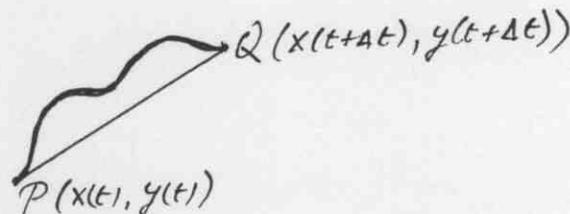
$$\frac{\text{length of the arc } PQ}{\text{length of the chord } PQ} \xrightarrow[\Delta t \to 0]{} 1,$$

where $P = (x(t), y(t))$, $Q = (x(t + \Delta t), y(t + \Delta t))$.

Let $s(t)$ denote the length of the arc from

$$(x(a), y(a)) \quad \text{to} \quad (x(t), y(t)).$$

Then our requirement is

$$\lim_{\Delta t \to 0} \frac{|\Delta s|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 1. \qquad \begin{aligned} \Delta s &= s(t + \Delta t) - s(t) \\ \Delta x &= x(t + \Delta t) - x(t) \\ \Delta y &= y(t + \Delta t) - y(t). \end{aligned}$$

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Hence

$$\lim_{\Delta t \to 0} \frac{\frac{\Delta s}{\Delta t}}{\sqrt{(\frac{\Delta x}{\Delta t})^2 + (\frac{\Delta y}{\Delta t})^2}} = 1.$$

Thus, since $\lim_{\Delta t \to 0} \sqrt{(\frac{\Delta x}{\Delta t})^2 + (\frac{\Delta y}{\Delta t})^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$, it follows that

$$\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \left[\frac{\frac{\Delta s}{\Delta t}}{\sqrt{(\frac{\Delta x}{\Delta t})^2 + (\frac{\Delta y}{\Delta t})^2}} \sqrt{(\frac{\Delta x}{\Delta t})^2 + (\frac{\Delta y}{\Delta t})^2} \right]$$

$$= 1 \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} .$$

Hence the length s of the curve is given by

$$s = s(b) - s(a) = \int_a^b \frac{ds}{dt} \, dt = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, dt.$$

The formula for the length s of the curve $x = x(t), y = y(t), a \le t \le b$, is

$$s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} \, dt.$$

This has an interesting dynamic interpretation. If the vector $r(t) = (x(t), y(t))$ is regarded as the position vector of a moving point at time t, then $v(t) = r'(t) = (x'(t), y'(t))$ is the velocity of the point at time t. Thus its magnitude $|v(t)| = \sqrt{x'(t)^2 + y'(t)^2}$ is the speed of the point at time t and the distance travelled between times $t = a$ and $t = b$ is

$$s = \int_a^b |v| = \int_a^b |\frac{dr}{dt}| dt.$$

We may also regard the arc length formula as an extension to general curves of Pythagoras' Theorem. Our basic requirement is
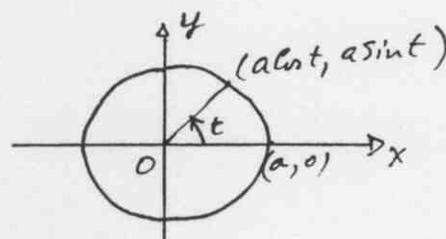
$$\Delta s \simeq \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\frac{\Delta x}{\Delta t})^2 + (\frac{\Delta y}{\Delta t})^2}\, \Delta t$$

or, formally, $\quad ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt.$
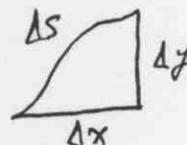
EXAMPLE 9.2.1: The curve $\quad x = a \cos t,$
$y = a \sin t, \quad 0 \le t \le 2\pi$ is the

circle of radius a

centered at $(0,0)$.

Its length is

$$S = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-a \sin t)^2 + (a \cos t)^2}\, dt$$

$$= \int_0^{2\pi} a\, dt = 2\pi a.$$

PROPOSITION 9.2.2. *If f is differentiable on $[a, b]$,*
the length of its graph is
$S = \int_a^b \sqrt{1 + f'(t)^2}\, dt.$

This graph has parametric representation

$$x = t, \quad y = f(t), \quad a \le t \le b.$$
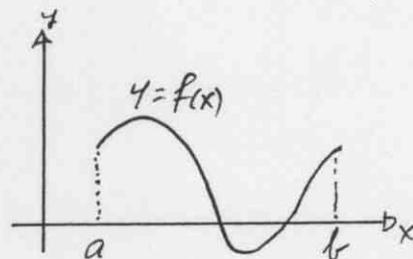
Thus

$$s = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt = \int \sqrt{1 + f'(t)^2}\, dt.$$

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EXAMPLE 9.2.3:

The length of the curve

$y = x^2$, $0 \le x \le 1$, is

$$s= \int_0^1 \sqrt{1+(\tfrac{dy}{dx})^2}\, dx = \int_0^1 \sqrt{1+4x^2}\, dx$$
$$= \int_0^\alpha \tfrac{1}{2} \sec^3 \theta\, d\theta$$

$x = \tfrac{1}{2} \tan \theta$
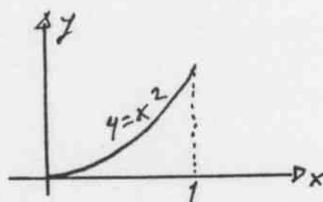$dx = \tfrac{1}{2} \sec^2 \theta d\theta$
$\sqrt{1+4x^2} = \sec \theta$
$0 \le x \le 1 \longleftrightarrow 0 \le \tan \theta \le 2$
$\alpha = \arctan 2$

$$= \tfrac{1}{2} \int_0^\alpha \sec \theta d(\tan \theta)$$

$$= \tfrac{1}{2} \sec \theta \tan \theta|_0^\alpha - \tfrac{1}{2} \int_0^\alpha \tan^2 \theta \sec \theta\, d\theta$$
$$= \tfrac{1}{2} \sec \theta \tan \theta|_0^\alpha - \tfrac{1}{2} \int_0^\alpha \sec^3 \theta\, d\theta + \tfrac{1}{2} \int_0^\alpha \sec \theta\, d\theta.$$

Thus

$$s = \tfrac{1}{4}[\sec \theta \tan \theta + \log|\sec \theta + \tan \theta|]_0^\alpha \qquad \tan \alpha = 2$$

$$= \tfrac{1}{4}[2\sqrt{5} + \log(\sqrt{5}+2)] \qquad \sec \alpha = \sqrt{5}$$

$$= \tfrac{\sqrt{5}}{2} + \tfrac{1}{4} \log(\sqrt{5}+2).$$
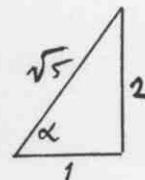
To address the problem of finding the area of a surface of revolution, first note (Problem 9.1) that the area of a trapezoid is

$$\frac{1}{2}(b_1 + b_2)h,$$
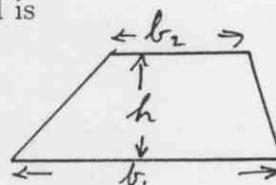
where b_1, b_2 are the lengths of the two parallel sides and h is the distance between them.

From this we find that

a portion of a conical

surface has area

$$\frac{1}{2}(2\pi r_1 + 2\pi r_2)\ell = \pi(r_1 + r_2)\ell$$

where r_1, r_2 are the radii of the bottom and top and ℓ is the 'slant height'.

The basic requirement which we impose on the area of the surface generated when the smooth curve $x = x(t), y = y(t), a \le t \le b$ is rotated about an axis in the same plane is that

$$\lim_{\Delta t \to 0} \frac{\text{Area generated by the arc } PQ}{\text{Area generated by the chord } PQ} = 1.$$

Let $S(t)$ denote the area generated by the arc from $(x(a), y(a))$ to $(x(t), y(t))$. Let $r(t)$ be the distance of $(x(t), y(t))$ from the axis of rotation.

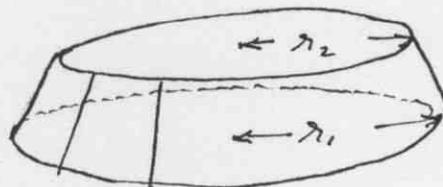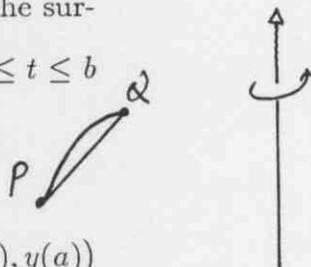Then, since the chord PQ generates a portion of a conical surface, we have

$$\lim_{\Delta t \to 0} \frac{|\Delta S|}{\pi[r(t) + r(t + \Delta t)]\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 1,$$

where

$$\Delta S = S(t + \Delta t) - S(t),$$

$$\Delta x = x(t + \Delta t) - x(t),$$

$$\Delta y = y(t + \Delta t) - y(t).$$

It follows that

$$\frac{dS}{dt} = 2\pi r(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

and hence the surface area S is

$$S = S(b) - S(a) = \int_{t=a}^{t=b} \frac{dS}{dt}\, dt = 2\pi \int_a^b r(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

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derivation of this is

$$dS = 2\pi r ds$$

$$= 2\pi r \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

Thus

$$S = 2\pi \int_a^b r(t)\ \sqrt{x'(t)^2 + y'(t)^2}\ dt$$

as before.

PROPOSITION 9.2.4. *The area generated by rotation of the curve*

$y = f(x), a \le x \le b,$ about

(a) *the   x-axis   is* $2\pi \int_a^b |f(x)| \sqrt{1 + f'(x)^2}\ dx,$

(b) *the   y-axis is   is* $2\pi \int_a^b |x| \sqrt{1 + f'(x)^2}\ dx$
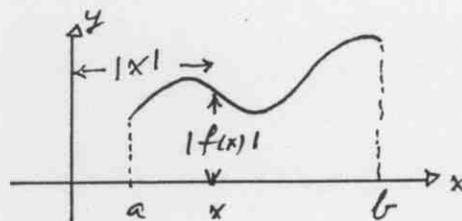
EXAMPLE 9.2.5: If we rotate the semicircle $x = a \cos t,$

$y = a \sin t,\quad 0 \le t \le \pi$ about the   x-axis   we generate a sphere of

radius a.   Here

$$r(t) = a \sin t \quad \text{so}$$

$$S = 2\pi \int_0^\pi r(t)\ \sqrt{x'(t)^2 + y'(t)^2}\ dt$$

$$= 2\pi \int_0^\pi a \sin t \cdot a\ dt$$

$$= -2\pi a^2 \cos t|_0^\pi = 4\pi a^2.$$
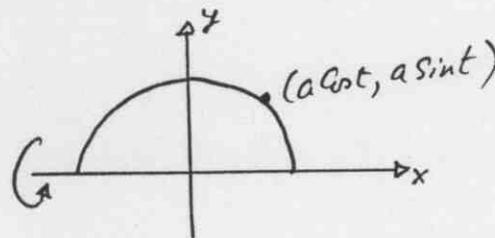
EXAMPLE 9.2.6: The surface area of the torus

generated by rotating a circle of radius a

about an axis in the plane of the circle

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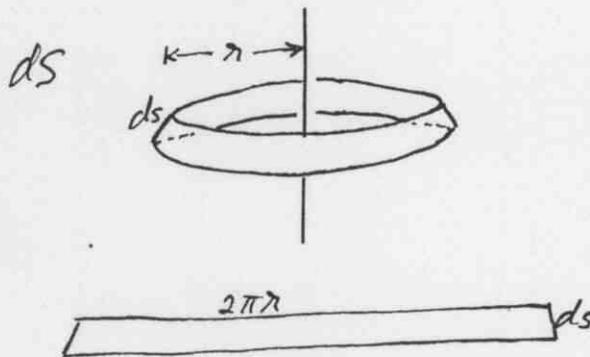
at a distance $b > a$ from its centre is $4\pi^2 ab$. Here, let
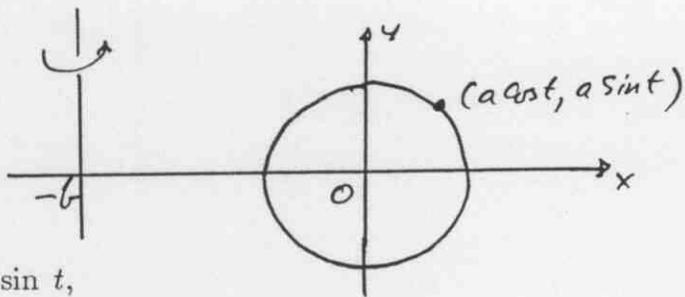
$$x = a \cos t, \quad y = a \sin t,$$
$$r(t) = b + a \cos t, \quad 0 \le t \le 2\pi.$$

Then

$$S = 2\pi \int_0^{2\pi} r(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$= 2\pi \int_0^{2p} (b + a \cos t) \cdot a\, dt$$

$$= 4\pi^2 ab, \quad \text{since} \quad \int_0^{2\pi} \cos t\, dt = 0.$$

§9.3. Optional Applications of Integration.

This section contains several further applications of integration. The motivation given for the representation of each of the concepts as an integral is quite brief and formal.

Work. If a constant force F moves its point of application through a distance s in its own direction, then the work done is defined to be

$$W = Fs.$$
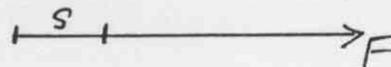
If force is measured in dynes and distance in centimeters, then work is measured in dyne-centimeters and a dyne-centimeter is called an erg. If force is measured in newtons and distance in meters the force is in newton-meters and a newton-meter is called a joule (1 joule $= 10^7$ ergs).

If the force is not constant but depends continuously on its position x, then the work done by the force in moving from $x = a$ to $x = b$ is defined to be

$$W = \int_a^b F(x)dx.$$

The motivation for this definition is that the work in moving from x to $x + \Delta x$ is approximately $\Delta W \simeq F(x)\Delta x$ so that we have $dW = F(x)dx$.

EXAMPLE 9.3.1: Hooke's Law asserts that the force needed to stretch a spring an amount x is kx where k is a constant of the spring, i.e. the restoring force in such a stretched spring is $-kx$. Suppose 2 joules are needed to extend a certain spring by 10 cm and we wish to determine the amount of work needed to stretch it from a 15 cm extension to a 25 cm extension. Here

$$W = \int_{15}^{25} kx \ dx = \frac{k}{2} \ x^2|_{15}^{25} = k200 \text{ joules.}$$

To determine k, the spring constant, we are given

$$2 = \int_0^{10} kx^2 dx = \frac{1}{2}kx^2|_0^{10} = k50$$

so that $k = 1/25$ and therefore

$$W = \frac{200}{25} \text{ joules } = 8 \text{ joules.}$$

EXAMPLE 9.3.2: The force (weight) acting on a body of mass m

at a distance r from the centre of the earth is

$$\frac{\gamma M m}{r^2}$$

where γ is a universal constant and M is the mass of the Earth. It is assumed that r exceeds the radius of the Earth. The work W necessary to lift the body from $r = a$ to $r = b$ is

$$W = \int_a^b \frac{\gamma M m}{r^2} \, dr = \gamma M m \left(\frac{1}{a} - \frac{1}{b}\right).$$

If the distances involved are small, it may be assumed that the weight of the body is constant.

EXAMPLE 9.3.3: A reservoir has the shape of a right circular cone with its vertex downwards. If the radius at the top is $4m$ and the depth is $10m$ find the work necessary to pump the full reservoir of water to a height $2m$ above the top.

Here
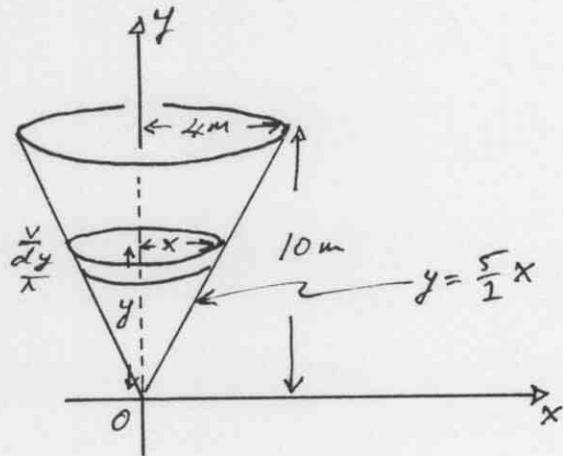
$$dW = \rho g \pi x^2 dy (12 - y)$$
$$= \rho g \pi \left(\frac{2}{5} y\right)^2 (12 - y) dy,$$

where $\rho = 1,000$ kg/m^3 is the density of water and $g = 9.81$ m/sec^2 is the acceleration due to gravity and dW is the work done in lifting

a thin 'slice' at a height y above the vertex. Thus

$$W = \int_0^{10} \rho g \pi (\frac{2}{5} y)^2 (12 - y) dy$$

$$= \frac{4}{25} \pi \rho g \int_0^{10} (12y^2 - y^3) dy$$

$$= \frac{4}{25} \pi \rho g (4y^3 - \frac{1}{4} y^4)|_0^{10}$$

$$= \frac{4}{25} \pi \rho g \left(4(10)^3 - \frac{1}{4}(10)^4\right)$$

$$= \frac{4}{25} \pi \rho g \, 10^3 (4 - \frac{5}{2})$$

$$= \frac{6}{25} \, 10^6 (9.81) \pi \quad \text{joules.}$$

Average Value. The *average value* $\bar{x}$ of n numbers $x_1, \ldots, x_n$ is defined to be

$$\bar{x} = \frac{1}{n}(x_1 + \cdots + x_n) = \frac{1}{n} \sum_{i=1}^n x_i.$$

For a function $f : [a, b] \to \mathbb{R}$ such that $\int_a^b f$ exists, the average value $\bar{f}$ of f on $[a, b]$ is defined to be

$$\bar{f} = \frac{1}{b - a} \int_a^b f(x) dx.$$

If f is a continuous function on $[a, b]$, then the Mean Value Theorem for integrals (Theorem 6.4.2) asserts that there exists a point $c \in [a, b]$ such that

$$\bar{f} = f(c).$$

EXAMPLE 9.3.4: The average value of the function x^2 on $[0, 4]$ is $\frac{1}{4} \int_0^4 x^2 dx = \frac{1}{4} \frac{4^3}{3} = \frac{16}{3}$.

The idea of average value may be extended to weighted averages
If $m_1, \ldots, m_n$ are positive numbers then the weighted average of
$x_1, \ldots, x_n$ with respect to these weights is

$$\bar{x} = (m_1 x_1 + \cdots + m_n x_n)/(m_1 + \cdots + m_n)$$
$$= (\sum_{i=1}^{n} m_i x_i)/(\sum_{i=1}^{n} m_i).$$

This concept is important in mechanics.
Suppose there are masses

$m_1, \ldots, m_n$ located

of $(x_1, y_1), \ldots, (x_n, y_n)$

respectively. Then, if
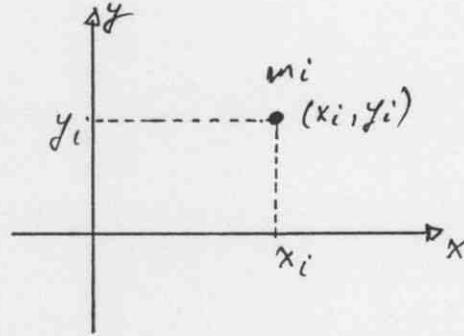
$$\bar{x} = (\sum_{i=1}^{n} m_i x_i)/(\sum_{i=1}^{n} m_i), \quad \bar{y} = (\sum_{i=1}^{n} m_i y_i)/(\sum_{i=1}^{n} m_i),$$

$(\bar{x}, \bar{y})$ is called the *centre of mass* of the system. Here $m_i x_i$ is
the moment of the i-th mass about the y-axis and $m_i y_i$ is its
moment about the x-axis. If $M = \sum_{i=1}^{m} m_i$,

$$M\bar{x} = \sum_{i=1}^{n} m_i x_i, \quad M\bar{y} = \sum_{i=1}^{n} m_i y_i$$

shows that the moments of the whole system about the x and y
axes are the same as those of a single mass $M = \sum_{i=1}^{m} m_i$ located at
$(\bar{x}, \bar{y})$.

EXAMPLE 9.3.5: The centre of mass

of a system $m_1 = 2, m_2 = 5$

located at $(1, 0), (2, 0)$

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respectively is $(\bar{x}, \bar{y})$ where
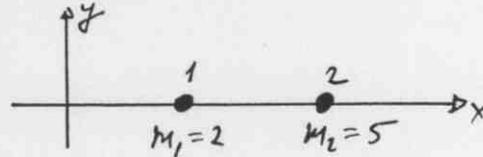
$$7\bar{x} = 2(1) + 5(2) \quad \text{(moment about } y\text{-axis)}$$

$$7\bar{y} = 2(0) + 5(0) \quad \text{(moment about } x\text{-axis)}.$$

Therefore $(\bar{x}, \bar{y}) = (\frac{12}{7}, 0)$.

EXAMPLE 9.3.6: (THIN ROD).

If $\rho(x) \geq 0$, $a \leq x \leq b$,

and $\int_a^b \rho$ exists, we

may interpret $\rho(x)$ to be the

density (mass/unit length) of a thin

rod S at $\{(x, 0) : a \leq x \leq b\}$ whose total mass is $M = \int_s dm \overset{\text{def}}{=} \int_1^b \rho(x)dx$. The centre of mass of the rod is $(\bar{x}, \bar{y})$ defined

by

$$\bar{x} = \frac{\int_S \tilde{x} dm}{\int_S dm} \overset{\text{def}}{=} \frac{\int_a^b x\rho(x)dx}{\int_a^b \rho(x)dx}$$

$$\bar{y} = \frac{\int_S \tilde{y} dm}{\int_S dm} \overset{\text{def}}{=} \frac{\int_a^b 0\rho(x)dx}{\int_a^b \rho(x)dx} = 0$$

and $M\bar{x} = \int_a^b xp(x)dx$, $M\bar{y} = 0$ are the moments about the y

and x axes respectively.

In the special case of a uniform rod $(\rho = \text{constant})$

$$\bar{x} = \frac{\int_a^b xdx}{\int_a^b dx} = \frac{\frac{1}{2}(b^2 - a^2)}{b - a} = \frac{1}{2}(b + a), \quad \bar{y} = 0$$

and the midpoint of the line segment, $(\frac{1}{2}(b + a), 0)$ is called its

centroid.

Suppose $C : x = x(t), y = y(t), a \le t \le b$ is a smooth curve

and $\rho(x, y) \ge 0$ is such that

$\int_C \rho\, ds \overset{\text{def}}{=} \int_a^b \rho(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt$ exists. We may interpret
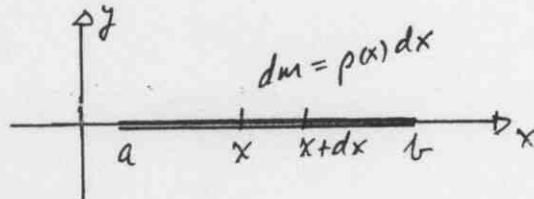
$\rho(x, y)$ to be the density (mass/unit length) at the point (x, y) of a

smooth wire in the shape C whose mass is $M = \int_C dm \overset{\text{def}}{=} \int_C \rho\, ds$.

The centre of mass of the wire is $(\bar{x}, \bar{y})$ defined by

$$\bar{x} = \frac{\int_C \tilde{x}\, dm}{\int_C dm} \overset{\text{def}}{=} \frac{\int_a^b x(t)\rho(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt}{\int_a^b \rho(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt}$$

$$\bar{y} = \frac{\int_C \tilde{y}\, dm}{\int_C dm} \overset{\text{def}}{=} \frac{\int_a^b y(t)\rho(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt}{\int_a^b \rho(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt}$$
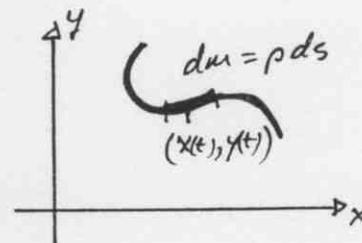
and $M\bar{x} \overset{\text{def}}{=} \int_C \tilde{x}\, dm, \quad M_y \overset{\text{def}}{=} \int_C \tilde{y}\, dm.$

For a uniform wire $(\rho = \text{constant})$

$$\bar{x} = \frac{\int_a^b x(t) \sqrt{x'(t)^2 + y'(t)^2}\, dt}{\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt}, \quad \bar{y} = \frac{\int_a^b y(t) \sqrt{x'(t)^2 + y'(t)^2}\, dt}{\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt}$$

and the point $(\bar{x}, \bar{y})$ is called the centroid of the curve C. Notice

that the value of the constant density ρ is irrelevant since it factors

out of the integrals and cancels.

EXAMPLE 9.3.7: For the semicircle

$C : x = a \cos t, \quad y = a \sin t, \quad 0 \le t \le \pi$

$ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt = a\, dt$

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and its centroid is $(\bar{x}, \bar{y})$ given by

$$\bar{x} = \frac{\int_C x\,ds}{\int_C ds} = \frac{\int_0^\pi a\cos t \cdot a\,dt}{\int_0^\pi a\,dt} = \frac{0}{\pi a} = 0$$

$$\bar{y} = \frac{\int_C y\,ds}{\int_C ds} = \frac{\int_0^\pi a\sin t \cdot a\,dt}{\pi a} = \frac{-a^2\cos t\big|_0^\pi}{\pi 0} = \frac{2a^2}{\pi a} = \frac{2a}{\pi}.$$
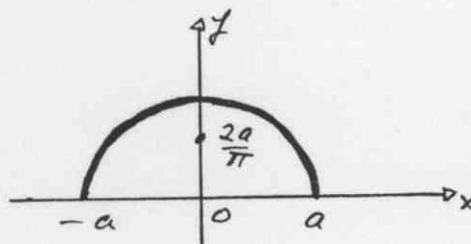
EXAMPLE 9.3.8:

The centroid of the

curve $y = f(x)$, $a \le x \le b$

is $(\bar{x}, \bar{y})$, when
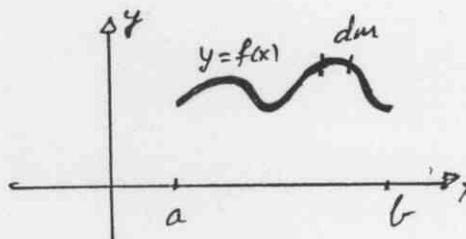
$$\bar{x} = \frac{\int_a^b x\,\sqrt{1 + f'(x)^2}\,dx}{\int_a^b \sqrt{1 + f'(x)^2}\,dx}, \quad \bar{y} = \frac{\int_a^b f(x)\,\sqrt{1 + f'(x)^2}\,dx}{\int_a^b \sqrt{1 + f'(x)^2}\,dx}.$$

For mass distributed over a plane area S of the type considered in § 9.1 we will mainly consider the case of constant density (mass/unit area) ρ and define the centroid of S.

Here we may disect
the region into 'thin
rods' of mass

$dm = \rho\,dA = \rho\ell(x)\,dx$

with centre of mass located

at midpoint of each rod $(\tilde{x}, \tilde{y})$. Note that $\tilde{x} = x$ and $\tilde{y}$ should
also be expressed as a function of x. Then
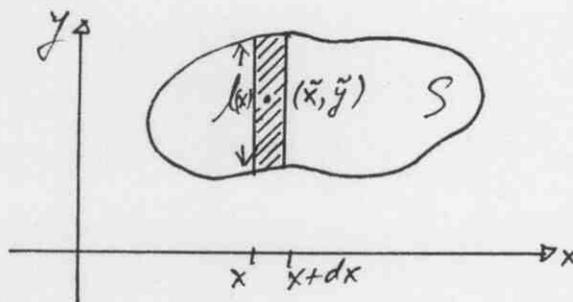
$$\bar{x} = \frac{\int_S \tilde{x}\,dm}{\int_S dm} = \frac{\int_a^b x\ell(x)\,dx}{\int_a^b \ell(x)\,dx}$$

$$\bar{y} = \frac{\int_S \tilde{y}\,dm}{\int_S dm} = \frac{\int_a^b \tilde{y}(x)\ell(x)\,dx}{\int_a^b \ell(x)\,dx}.$$

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We may take $\rho = 1$ if it is constant.

EXAMPLE 9.3.9:

The centroid of the

half disc

$$S = \{(x,y):\ 0 \le y \le \sqrt{a^2 - x^2},\ -0 \le x \le a\}$$

is $(\bar{x}, \bar{y}) = (0, \frac{4a}{3\pi})$. Disecting the region into 'rods' perpendicular

to the x-axis, we find
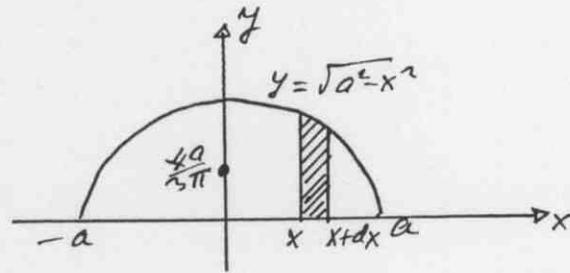
$$\bar{x} = \frac{\int_S \tilde{x}\,dm}{\int_S dm} = \frac{\int_{-a}^a x\sqrt{a^2 - x^2}\,dx}{\int_{-a}^a \sqrt{a^2 - x^2}\,dx} = \frac{0}{\frac{1}{2}\pi a^2} = 0$$

$$\bar{y} = \frac{\int_S \tilde{y}\,dm}{\int_S dm} = \frac{\int_{-a}^a \frac{1}{2}(a^2 - x^2)\,dx}{\int_{-a}^a \sqrt{a^2 - x^2}\,dx} = \frac{\frac{1}{2}a^2 x - \frac{1}{6}x^3\,|_{-a}^a}{\frac{1}{2}\pi a^2}$$

$$= \frac{a^3 - \frac{a^3}{3}}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi}\ .$$

Of course we should

have avoided the

calculation of $\bar{x}$ and

appealed to symmetry

to conclude $\bar{x} = 0$.
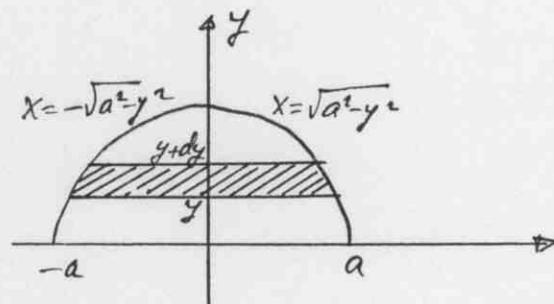
We could also find $\bar{y}$ from disections parallel to the x-axis

$$\bar{y} = \frac{\int_S \tilde{y}\,dm}{\int_S dm} = \frac{2\int_0^a y\sqrt{a^2 - y^2}\,dy}{2\int_0^a \sqrt{a^2 - y^2}\,dy} = \frac{-\frac{2}{3}(a^2 - y^2)^{3/2}\,|_0^a}{\frac{1}{2}\pi a^2}$$

$$= \frac{\frac{2}{3}a^3}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi}\ .$$

We could even use the result of Example 9.3.7

and disect the region

into semicirular 'wires'

of mass

$$dm = \pi x dx$$

centroid $(\tilde{x}, \tilde{y}) = (0, \frac{2x}{\pi})$ so that
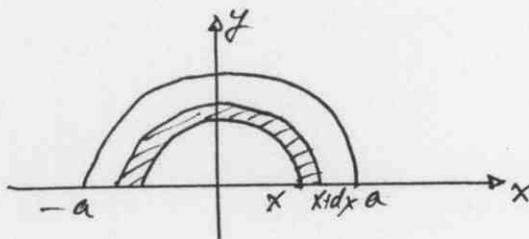
$$\bar{x} = 0, \quad \bar{y} = \frac{\int_S \tilde{y} dm}{\int_S dm} = \frac{\int_0^a \frac{2x}{\pi} \pi x dx}{\int_0^a \pi x dx} = \frac{\frac{2}{3} a^3}{\frac{1}{2} \pi a^2}$$

$$= \frac{4a}{3\pi} .$$

§9.4. Pappus' Theorems.

The First Theorem of Pappus states that if a surface S is generated by rotating a plane curve C through an angle α about an axis in the plane of the curve, then the area of the surface is given by the formula

(Area of S) = (Length of the path of the centroid of C)× (Length of C).

To see this, observe that the

area A is

$$A = \int_C \alpha x \, ds = \alpha \int_C x \, ds$$
$$= \alpha \bar{x} \int_C ds = \alpha \bar{x} s,$$
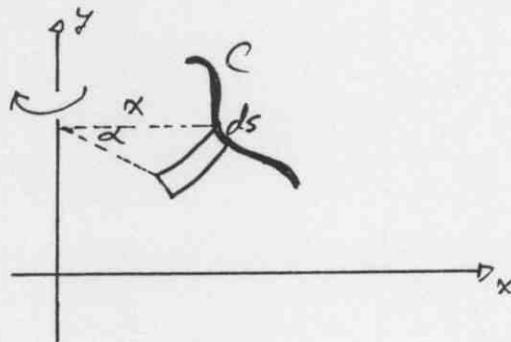
where S is the length of C, since $\bar{x} = \int_C x ds / \int_C ds$.

The Second Theorem of Pappus asserts that if a solid K is generated by rotating a plane area S about an axis in the plane,

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then the volume of K is given by the formula

(Volume of K) = (Length of the path of the centroid of S)$\times$ (Area of S).

By applying the method of
shells, we find that the
volume is

$$V = \int_a^b \alpha x \ell(x)\,dx = \alpha \int_a^b x\ell(x)\,dx$$
$$= \alpha\bar{x} \int_a^b \ell(x)\,dx = \alpha\bar{x}A$$
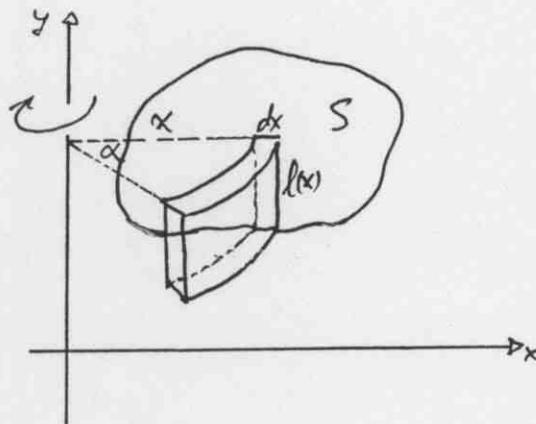
where A is the area of S,
since $\bar{x} = \int_a^b x\ell(x)\,dx / \int_a^b \ell(x)\,dx$.

For example, the surface area and volume of the torus generated
by rotating a circular disc of radius a about an axis in the plane of
the disc at a distance $b > a$ from its centre are

$$S = (2\pi b)(2\pi a)$$

(see Example 9.2.6)

$$V = (2\pi b)(\pi a^2)$$

(see Example 9.1.10).
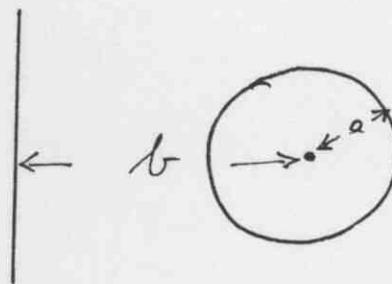
The result of Example 9.3.7 for the location of the centroid of a
semicircular curve may be deduced from the formula for the surface
area of the sphere it may be used to generate by means of Pappus'
First Theorem:

$$4\pi a^2 = (2\pi\bar{y})(\pi a) = 2\pi^2 a\bar{y}$$

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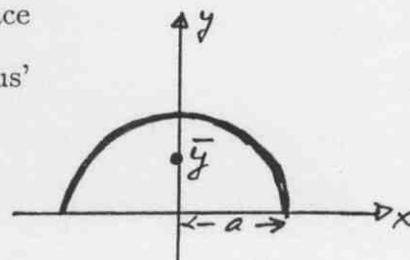
gives

$$\bar{y} = \frac{2a}{\pi}.$$

Pappus' Second Theorem may be used similarly (Problem 9.31) to locate the centroid of a semicircular area.

§9.5. Areas by Polar Coordinates.

In some situations, it is more convenient to use *polar coordinates* (r, θ) instead of the usual rectangular or cartesian coordinates (x, y) : these are related by

$$x = r \, \cos \, \theta$$

$$y = r \, \sin \, \theta$$

$$r \geq 0.$$
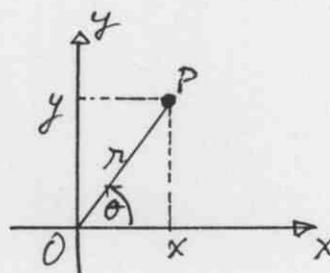
The polar coordinates of a point are not unique since

$$x = r \, \cos(\theta + 2m\pi) \quad y = r \, \sin(\theta + 2m\pi)$$

$m = 0, \pm 1, \pm 2, \ldots$ so that the

coordinates $(r, \theta + 2m\pi)$

specify the same point.

Moreover, since

$$r \, \cos(\theta + \pi) = -r \, \cos \, \theta$$

$$r \, \sin(\theta + \pi) = -r \, \sin \, \theta,$$

the polar coordinates

$(r, \theta + \pi), \quad (-r, \theta)$

also represent the same point, if

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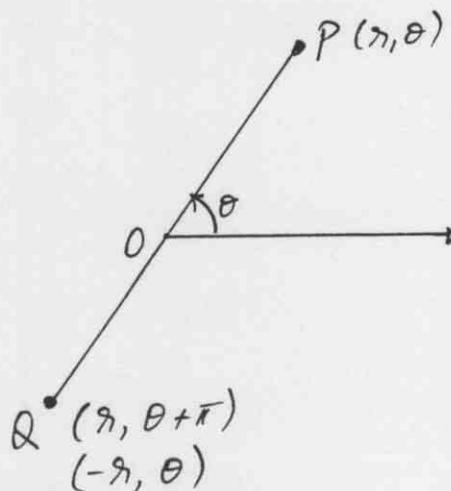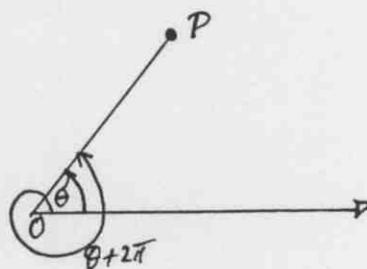
we allow negative values of r.

In rectangular coordinates, the curves $x = c$, $y = k$ are straight lines parallel to the coordinate axes. In polar coordinates, the equations $r = c$, $\theta = k$ represent circles centered on 0 and straight $\frac{1}{2}$-lines (rays) through 0, respectively.
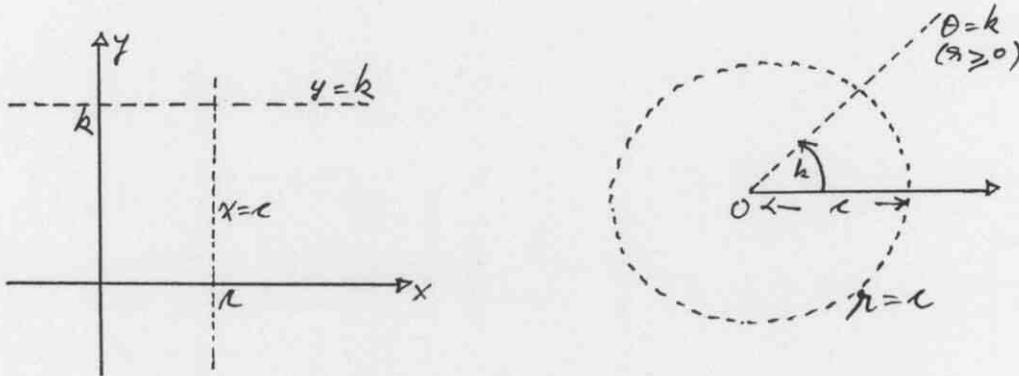
In practice it is convenient to allow negative values of r when using polar coordinates.

The equation of the straight line $x = c$ in polar coordinates is $r = c \sec \theta$, since $r \cos \theta = c$. Similarly $y = k$ is $r = k \operatorname{cosec} \theta$.

The equation

$$r \cos(\theta - \alpha) = p$$

represents a straight line L at a distance $|p|$ from 0 where the line

through 0 perpendicular to L makes an angle α with the x-axis. This can be seen from

$$r(\cos \alpha \cos \theta + \sin \alpha \sin \theta) = p$$

$$(\cos \alpha)x + (\sin \alpha)y = p.$$

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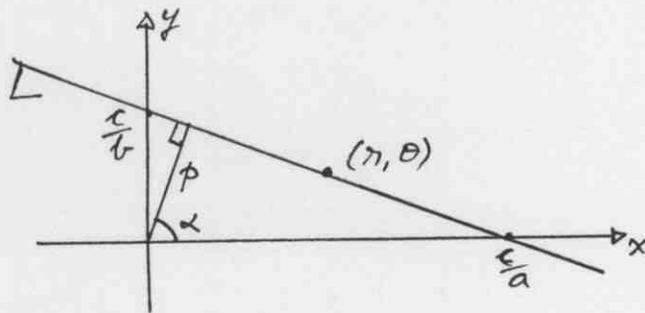
This is the same line as

$$ax + by = c$$

where $\cos \alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\sin \alpha = \frac{b}{\sqrt{a^2+b^2}}$, $p = \frac{c}{\sqrt{a^2+b^2}}$.

The distance between the points $(r_1, \theta_1), (r_2, \theta_2)$ is given by the Cosine Law (Problem 7.41)
$$d^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1).$$
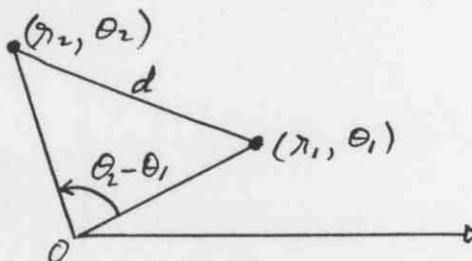
EXAMPLE 9.5.1: To find an equation for the circle centered at $(a, 0)$, radius $a > 0$, in polar coordinates, note that this curve has cartesian equation
$$(x - a)^2 + y^2 = a^2, \quad \text{or}$$
$$x^2 + y^2 - 2ax = 0.$$
Thus $r^2 - 2ar \cos \theta = 0$
$$r(r - 2a \cos \theta) = 0;$$

Since $r = 0$ represents only the point 0, the curve is given by $r = 2a \cos \theta$. The upper half of the circle is given by $r = 2a \cos \theta$, $0 \le \theta \le \frac{\pi}{2}$, and the lower half by $r = 2a \cos \theta$, $\frac{\pi}{2} \le \theta \le \pi$, in which case r is negative. Thus the complete circle is given by

$$r = 2a \cos \theta, \quad 0 \le \theta \le \pi.$$

It is also given by

$$r = 2a \cos \theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

in which case $r \ge 0$ for all θ.

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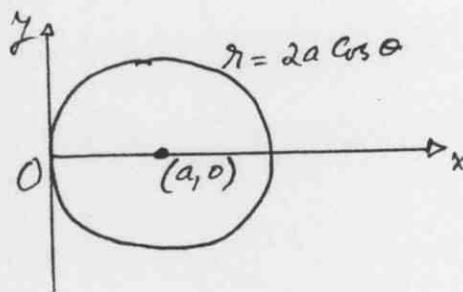
If f is continuous on $[\alpha, \beta]$, $f \geq 0$ and $\beta - \alpha \leq 2\pi$, then

the area of the region

$$\{(r, \theta) : 0 \leq r \leq f(\theta), \; \alpha \leq \theta \leq \beta\}$$

is

$$A = \tfrac{1}{2} \int_\alpha^\beta f(\theta)^2 \, d\theta.$$

This formula comes from

an estimate of the form

$$\tfrac{1}{2} f(\theta_*)^2 \Delta\theta \leq \Delta A \leq \tfrac{1}{2} f(\theta^*)^2 \Delta\theta$$

which implies

$$\tfrac{dA}{d\theta} = \tfrac{1}{2} f(\theta)^2.$$

The argument may be formally summarized as

$$dA = \tfrac{1}{2} r^2 d\theta, \quad r = f(\theta).$$

More generally, the area of the region

$$\{(r, \theta) : \; g(\theta) \leq r \leq f(\theta), \quad \alpha \leq \theta \leq \beta\}$$

is

$$A = \tfrac{1}{2} \int_\alpha^\beta [f(\theta)^2 - g(\theta)^2] d\theta.$$
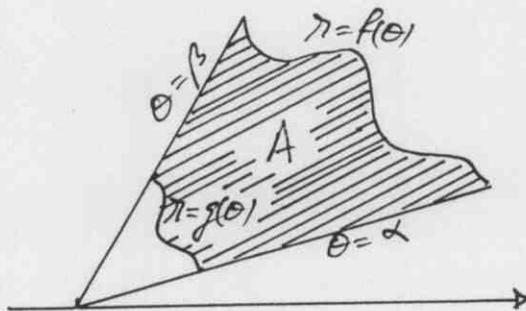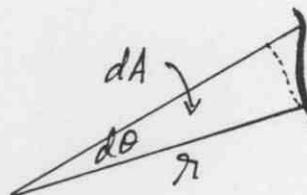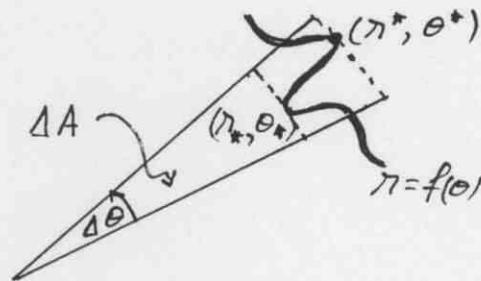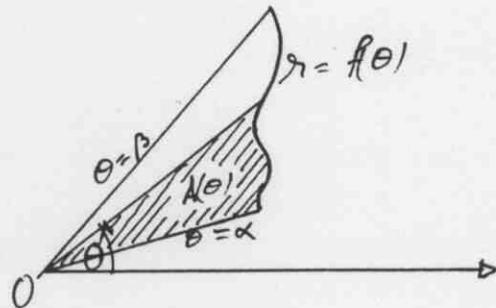
EXAMPLE 9.5.2: The area of one of the crescent

shaped regions determined

by two circles of radius

a each passing through

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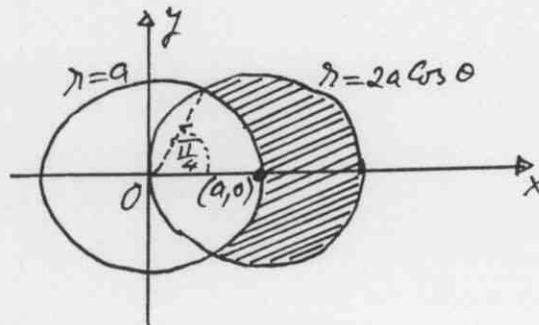
the centre of the other is

$$A = \frac{1}{2} \int_{-\pi/3}^{\pi/3} (4a^2 \, \cos^2 \theta - a^2) d\theta$$

$$= a^2 \int_0^{\pi/3} (4 \, \cos^2 \theta - 1) d\theta$$

$$= a^2 \int_0^{\pi/3} [2(1 + \, \cos \, 2\theta) - 1] d\theta$$

$$= a^2 \int_0^{\pi/3} (1 + 2 \, \cos \, 2\theta) d\theta$$

$$= a^2 (\theta + \sin \, 2\theta)|_0^{\pi/3}$$

$$= a^2 (\frac{\pi}{3} + \frac{\sqrt{3}}{2}).$$

EXAMPLE 9.5.3: The curve $\quad r = a(1 + \cos \, \theta),$

the heart-shaped

curve on the right, is

called a *cardioid*. The

area which it encloses is
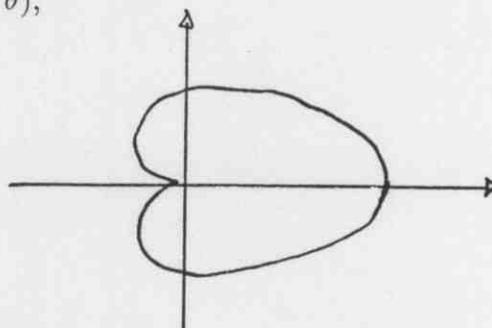
$$A = \frac{1}{2} \int_{-\pi}^{\pi} a^2 (1 + \cos \, \theta)^2 d\theta$$

$$= \frac{a^2}{2} \int_{-\pi}^{\pi} (1 + 2 \, \cos \, \theta + \cos^2 \theta) d\theta$$

$$= \frac{a^2}{2} \int_{-\pi}^{\pi} [1 + 2 \, \cos \, \theta + \frac{1}{2} (1 + \, \cos \, 2\theta)] d\theta$$

$$= \frac{3\pi a^2}{2}.$$

Problems

9.1 Show that the area of a trapezoid is $\frac{1}{2}(b_1+b_2)h$, where b_1, b_2 are the lengths of the parallel sides and h is the distance between them.

9.2. Show that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

9.3. Show that the area of the region

$$\{(x,y) : p \le x \le q,\ x^2 + y^2 \le a^2\}$$

is

$$a^2 [\arcsin\left(\frac{q}{a}\right) - \arcsin\left(\frac{p}{a}\right)] + q\sqrt{a^2 - q^2} - p\sqrt{a^2 - p^2}\ .$$
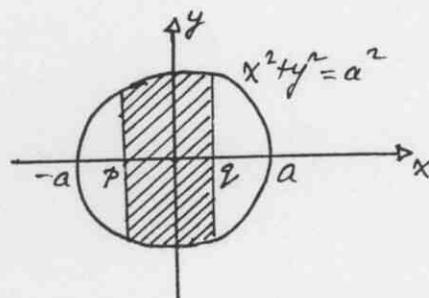
9.4. The area bounded by the curve $y = x^2$, $y = 4$ is $\frac{32}{3}$. Show this is 2 ways: by slicing the region into strips (a) perpendicular to the x-axis and (b) parallel to the x-axis.

9.5. The point $P = (\cosh t, \sinh t)$ lies on the branch of the hyperbola $x^2 - y^2 = 1$ for which $x > 0$. If $0 = (0,0)$, $Q = (1,0)$, show that the area of the curvilinear triangle formed by the line segments OP, OQ and the arc PQ of the hyperbola is $\frac{|t|}{2}$.

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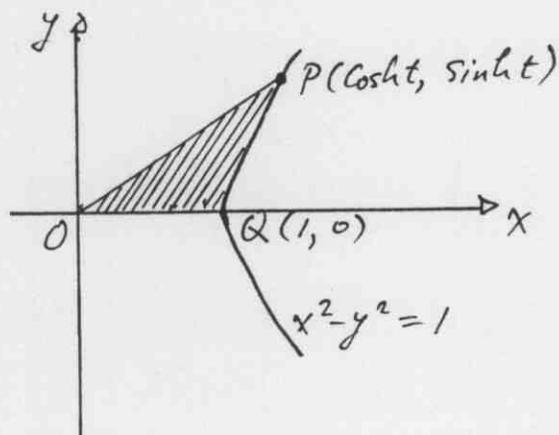
9.6. The area bounded by the curves $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$. Find a.

9.7. A solid has a circular base of radius 2. Each cross section perpendicular to a fixed diameter of the base is an equilateral triangle. Show that its volume is $\frac{32}{\sqrt{3}}$.
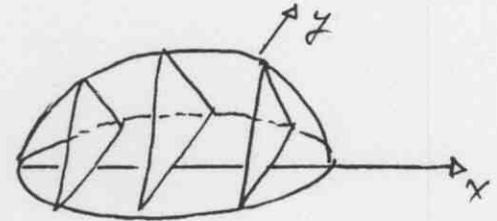
9.8. A wedge is cut from the base of a right circular cylinder of radius a by a plane through a diameter of the base and inclined at an angle α to the base. Show that the volume of the wedge is $\frac{2a^3}{3} \tan \alpha$.
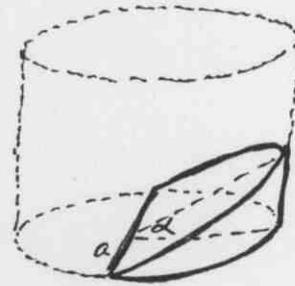
9.9. The axes of 2 solid right circular cylinders, each of radius a intersect at right angles. Show that the volume of their intersection is $\frac{16}{3} a^3$.

This is $\frac{1}{8}$ of the volume.
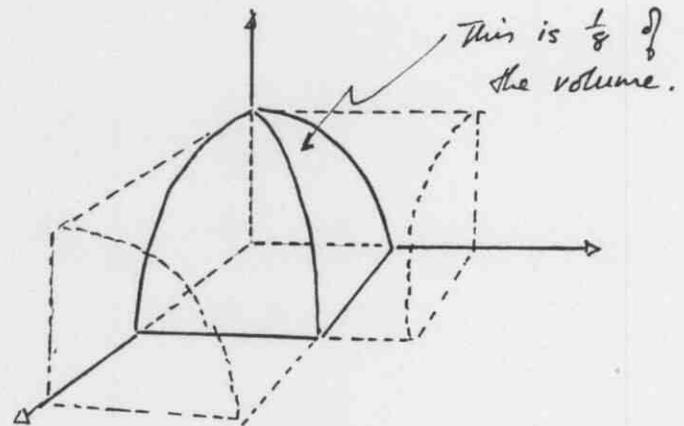
9.10. A cylindrical hole of length h is drilled through a sphere with the axis of the cylinder

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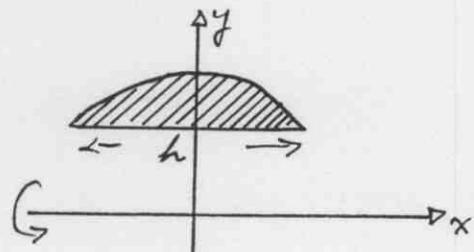
being a diameter of the sphere.

Show that the volume of the remaining solid is $\frac{\pi}{6} h^3$.

9.11. Show that the length of the curve $y = \log x$, $\sqrt{3} \leq x \leq \sqrt{8}$
is $1 + \log \sqrt{\frac{3}{2}}$.

9.12. The position of a particle at time t is given by $x = a \cos t + at \sin t$, $y = a \sin t - at \cos t$. Show that the distance travelled between $t = 0$ and $t = \frac{\pi}{2}$ is $\frac{\pi^2 a}{8}$.

9.13. Show that the volume of the solid generated by rotating one arch of the curve $y = \sin x$ about the x-axis is $\frac{\pi^2}{2}$.

9.14. Sketch the curve $x^{2/3} + y^{2/3} = a^{2/3}$ $(a > 0)$. This curve is called an *astroid*. Show that its length is $6a$. HINT: Try to find a parametric representation for it. If $x^{1/3} = X$, $y^{1/3} = Y$, what is a good parametric representation for X, Y?

9.15. Show that the centroid of the region bounded by the x-axis and the curve $y = c^2 - x^2$ is located at the point $(0, \frac{2}{5} c^2)$.

9.16. A spring has natural length 1 meter; a force of 100 Newtons compresses it to 0.9 m. How many joules of work are required to compress it to half its natural length? What is the length of the spring when 20 joules of work have been expended?

9.17. A tank has the shape of a paraboloid of revolution. The radius of the top is 4 m and its depth is 10 m. If the tank is full of water, find the work necessary to overcome gravity in pumpting all of the water out at the top. HINT: The paraboloid is generated by rotating a parabola $y = kx^2$ about the y-axis. First

determine k, then 'slice' the water into discs.

9.18. Show that the centroid of the region in the first quadrant which is bounded by the coordinate axes and the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is located at $\left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$.

9.19. Show that the centroid of the shaded region in the diagram is $\left(-\frac{a}{3}, \frac{28a}{9\pi}\right)$. The curves are semicircles. No integration is needed here.
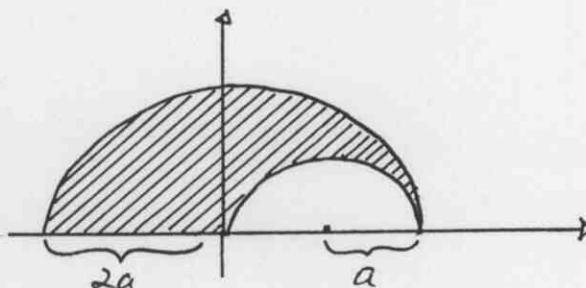
9.20. Show that the centroid of a triangle is located at a height $\frac{1}{3}h$ above the base, where h is the height of the triangle.
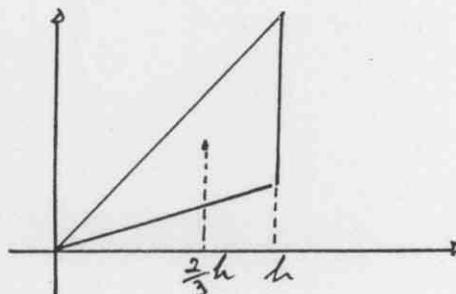
9.21. Determine the height of the centroid of a solid cone above its base.
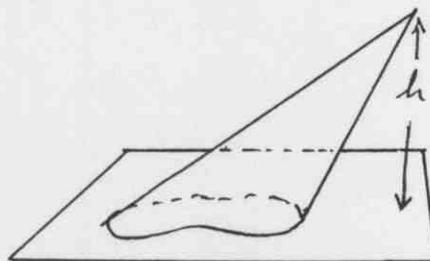
9.22. Find the volume of the following horn-shaped solid: its cross-section by any plane perpendicular to the x-axis, $0 \le x \le 4$, is a circular disc where diameter in the (x, y)-plane is the line segment joining the points $(x, \sqrt{x}/2)$, $(x, \sqrt{x})$.

9.23. A hemispherical bowl has radius a cm. If water is pouring into the bowl at a rate of k cm^3/sec find the rate at which

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the level of water, in the bowl is rising when its depth is h cm, $0 \le h < a$.

9.24. What is the area of
the shaded region determined
by the three circles?
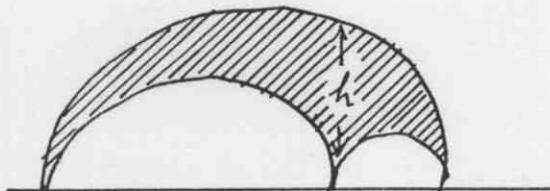
9.25. A heavy buoy in the
shape of a right circular
cone of weight w floats in
a lake to a depth h
(vertex down). The buoy is
raised by a winch until it is just
clear of the water. Show that the work done in overcoming
gravity is $\frac{3}{4} wh$.
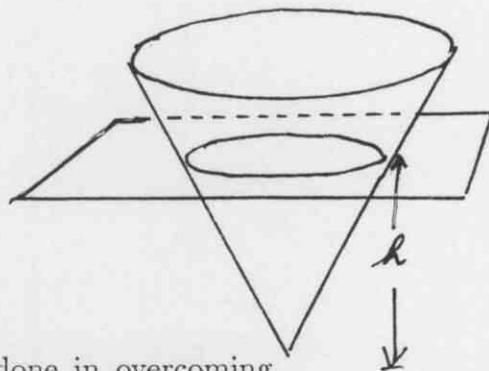
9.26. Show that the area enclosed by one loop of the lemniscale $r^2 = a^2 \cos 2\theta$ is $\frac{a^2}{2}$.

9.27. Show that the area of one leaf of the rose $r = a \cos 3\theta$ is $\pi a^2/12$.

9.28. Show that the area inside the cardioid $r = a(1 + \cos \theta)$ and outside the circle $r = a$ is $a^2(8 + \pi)/4$.

9.29. Consider the curve $r = \frac{1}{2} + \cos \theta$. Show that the area enclosed by the large loop is $\frac{\pi}{2} + \frac{3\sqrt{3}}{8}$ and by the small loop is $\frac{\pi}{4} - \frac{3\sqrt{3}}{8}$.

9.30. Show that centroid of the plane region enclosed by the curves

$r = f(\theta)$, $\theta = \alpha$, $\theta = \beta$ $(f \geq 0,\ 0 \leq \alpha < \beta \leq 2\pi)$, is given by

$$A\bar{x} = \frac{1}{3} \int_\alpha^\beta r^3 \cos \theta d\theta \quad A\bar{y} = \frac{1}{3} \int_\alpha^\beta r^3 \sin \theta d\theta$$

where $A = \frac{1}{2} \int_\alpha^\beta r^2 d\theta$, $r = f(\theta)$.

9.31. Use Pappus' Second Theorem to show that the centroid of a semicircular disc of radius 0 is located at a distance $\frac{4a}{3\pi}$ above its base.
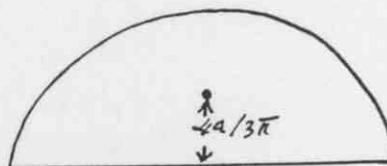
9.32. Show that the equation

$$r^2 - 2rp \cos(\theta - \alpha) + p^2 = a^2$$

represents a circle with centre (p, α) and radius a in polar coordinates.

9.33. Find a formula for the length of the curve whose equation in polar coordinates is $r = f(\theta)$, $\alpha \leq \theta \leq \beta$.

9.34. If the region $\{(r, \theta) : 0 \leq r \leq f(\theta),\ \alpha \leq \theta \leq \beta\}$ is rotated about the line $\theta = 0$, find a formula for the volume generated.

9.35. A circle of radius a rolls along the x-axis. A point P on the circle traces a *cycloid*. Let θ denote the angle through which

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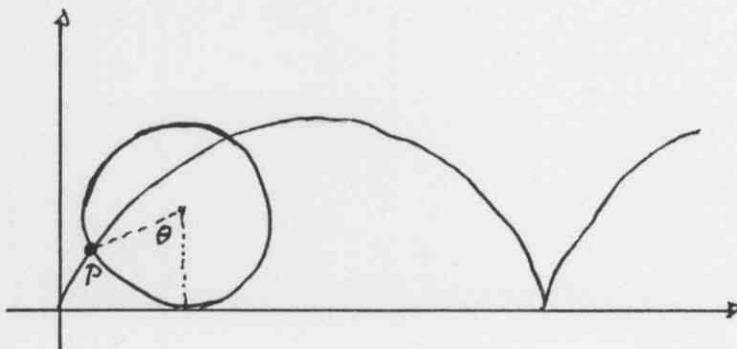
the wheel has rotated.

(a) If P is at the origin

initially, show

$$x = a(\theta - \sin \theta)$$

$$y = a(1 - \cos \theta).$$
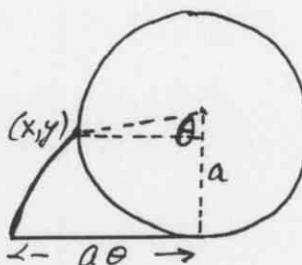
(b) Show that the length of one arch of the cycloid is $8a$.

9.36. The *folium of*

Descartes has equation

$$x^3 + y^3 = 3axy$$

(a) By considering $t = \frac{y}{x}$,

show that the folium

may be represented parametrically

by $x = \frac{3at}{1+t^3}$, $y = \frac{3at^2}{1+t^3}$.

(b) Use (a) to show that the line $x + y = -a$ is an

asymptote by showing $x + y \to -a$ as $t \to -1$.

(c) Since the curve is unchanged if we swap x and y, it

is symmetric about the line $y = x$. Use this information,

the geometric meaning of t and (a), (b) to verify as

much as you can about the nature of the curve.

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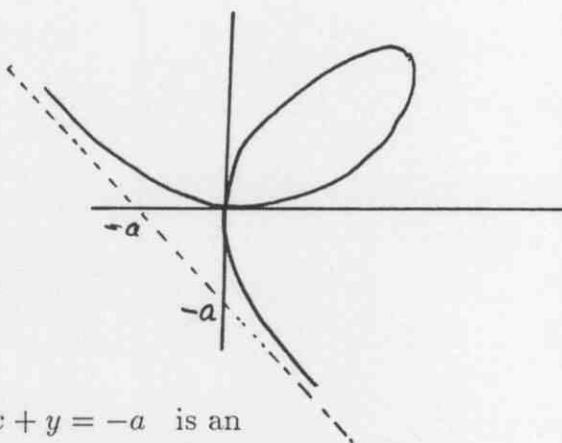
X. APPROXIMATION

§10.1. Taylor's Theorem.

Let f be a function such that $f(a), f'(a), \ldots, f^{(n)}(a)$ exist. Then the *Taylor Polynomial* p_n of f at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Notice that

$$p_n^{(k)}(a) = f^{(k)}(a), \quad k = 0, \ldots, n;$$

thus the functions f, p_n agree in their values and the values of their derivatives up through the n-th derivative at the point a.

$$p_0(x) = f(a)$$
$$r_0(x) = f(x) - p_0(x)$$
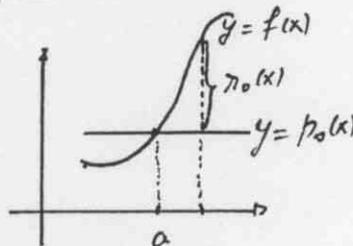
$$p_1(x) = f(a) + \frac{f'(a)}{1!}(x-a)$$
$$r_1(x) = f(x) - p_1(x).$$
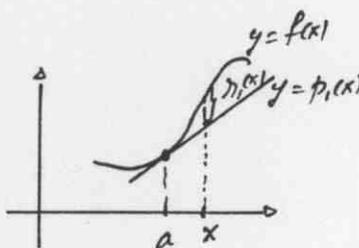
$$p_2(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$$
$$r_2(x) = f(x) - p_2(x).$$
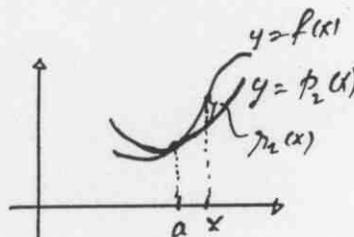
For 'nice' functions $p_n(x)$ should be close to $f(x)$ especially when x is near a. The 'error' in approximating $f(x)$ by $p_n(x)$ is

$r_n(x) = f(x) - p_n(x)$. Taylor's Theorem is concerned with estimating the remainder $r_n(x)$.

THEOREM 10.1.1. (TAYLOR'S THEOREM).

(a) If $f^{(n+1)}$ exists on $[a, x]$, $x > a$ (or $[x, a]$, $x < a$), then

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1} \quad (Lagrange\ form)$$

for some $c \in (a, x)$ (or $c \in (x, a)$).

(b) If $f^{(n+1)}$ is continuous on $[a, x]$, $x > a$ (or $[x, a]$, $x < a$), then

$$r_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x - t)^n dt \quad (Integral\ form).$$

Here

$$r_n(x) = f(x) - p_n(x) \quad \text{and}$$

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k.$$

We have already encountered both forms (a) and (b) of Taylor's Theorem in the case $n = 0$. If f' exists on $[a, x]$ (or $[x, a]$), then the Mean Value Theorem asserts that

$$f(x) = f(a) + f'(c)(x - a),$$

which is the statement $r_0(x) = \frac{f'(c)}{1!} (x - a)$.

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If f' is continuous on $[a, x]$ (or $[x, a]$), then the Fundamental Theorem of Calculus states that

$$f(x) = f(a) + \int_a^x f'(t)dt,$$

or equivalently $r_0(x) = \frac{1}{0!} \int_a^x f'(t)(x - t)^0 dt.$

Two other forms of the remainder are

$$r_n(x) = \frac{f^{(n+1)}(c)}{n!} (x - a)(x - c)^n \quad (Cauchy\ form)$$

$$r_n(x) = \frac{f^{(n+1)}(c)}{n!m} (x - a)^m (x - c)^{n-m+1} \quad (Schl\ddot{o}milch\ form)$$

In each case c is some numbers between a and x and, in the Schlömilch form, $m \geq 1$. The Cauchy and Lagrange forms of the remainder r_n are the cases $m = 1$ and $m = n+1$ respectively of the Schlömilch form.

We postpone proving Taylor's Theorem until after we have discussed some of its implications.

EXAMPLE 10.1.2: If f is a polynomial of degree n then $f^{n+1}(x) = 0$ for all x so that Taylor's Theorem may be used to express $f(x)$ in powers of $(x - a)$:

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k.$$

For example, if $f(x) = (1 + x)^n$, then $f(0) = 1$, $f^{(k)}(0) = n(n - 1)\ldots(n - k + 1)$, $k = 1, \ldots, n$ and $f^{(k)}(0) = 0$, $k > n$, so that

$$(1 + x)^n = 1 + nx + \frac{n(n - 1)}{1.2} x^2 + \cdots + x^n \quad (Binomial\ Theorem).$$

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EXAMPLE 10.1.3: If $f(x) = x^{1/2}$, then for $x > 0$,

$$f'(x) = \frac{1}{2}\, x^{-1/2}, \quad f^{(2)}(x) = -\frac{1}{4}\, x^{-3/2}, \quad f^{(3)}(x) = \frac{3}{8}\, x^{-5/2} \, .$$

Let $x = 98$, $a = 100$, $n = 2$; then the Lagrange form gives

$$98^{1/2} = 100^{1/2} + \frac{1}{1!}\, \frac{1}{2}\, 100^{-1/2}(98 - 100)$$

$$+ \frac{1}{2!}(-\frac{1}{4})100^{-3/2}(98 - 100)^2 + r_2(98)$$

$$= 10 + \frac{1}{2}(0.1)(-2) - \frac{1}{8}(0.001)(-2)^2 + r_2(98)$$

$$= 9.8995 + r_2(98)$$

where $r_2(98) = \frac{1}{3!}\, \frac{3}{8}\, c^{-5/2}(-2)^3 = -\frac{1}{2}\, c^{-5/2}$ for some $c \in$ (98, 100). The fact that $r_2(98) < 0$ implies $98^{1/2} < 9.8995$. A better estimate follows from $98 < c < 100$ so that $81 < c <$ 100 and hence $9 < c^{1/2} < 10$, $10^{-5} < c^{-5/2} < 9^{-5}$, $-\frac{1}{2.9^5} <$ $-\frac{1}{2}\, c^{-5/2} < -\frac{1}{2.10^5}$

$$-0.000009 < r_2(98) < -0.000005$$

and therefore

$$9.899491 < 98^{1/2} < 9.899495.$$

EXAMPLE 10.1.4: If $f(x) = e^x$, then $f^{(k)}(x) = e^x$, $k = 0, 1, 2, \ldots.$ Thus, with $a = 0$,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + r_n(x)$$

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where

$$r_n(x) = e^c \frac{x^{n+1}}{(n+1)!},$$

for some c between 0 and x. If $x > 0$, then since

$1 < e^c < e^x$

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n+1}}{(n+1)!} < e^x < 1 + \frac{x}{1!} + \cdots + \frac{x^n}{n!} + e^x \frac{x^{n+1}}{(n+1)!}.$$

If $x < 0$, then $r_n(x) < 0$ if n is even and $r_n(x) > 0$ if n is odd. For example,

$$e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + r_4 = 2.708\dot{3} + r_4$$

$$r_4 = \frac{e^c}{5!} \quad \text{for some} \quad c \in (0,1).$$

Thus $\frac{1}{5!} < r_4 < \frac{e}{5!} < \frac{3}{5!}$ (since $e < 3$),

$$0.008\dot{3} < r_4 < 0.025$$

so that $2.708\dot{3} + 0.008\dot{3} < e < 2.708\dot{3} + 0.025$, $2.71\dot{6} < e < 2.7\dot{3}$.

As an important application of Taylor's Theorem we show that e *is irrational*. Since $r_n(1) = e^c \frac{x^{n+1}}{(n+1)!}$ and $0 < c < 1$, we have

$$\frac{1}{(n+1)!} < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{3}{(n+1)!}, \quad 1 < e^c < e < 3.$$

Multiplying by $n!$,

$$\frac{1}{n+1} < n!e - \sum_{k=0}^{n} \frac{n!}{k!} < \frac{3}{n+1} \le \frac{3}{4} \quad n \ge 3.$$

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Now the sum is an integer for every n. If e were a rational number then $n!e$ is an integer for n sufficiently large and thus $n!e - \sum_{k=0}^{n} \frac{n!}{k!}$ is an integer between $\frac{1}{n+1}$ and $\frac{3}{4}$. But these two numbers are in the open interval $(0,1)$ and this contradiction shows that e is not rational.

As another application of Taylor's Theorem for the function e^x, consider $\int_0^1 e^{-t^2} dt$. Now

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{c^3}{3!} + \frac{x^4}{4!} + r_4(x), \quad r_4(x) = e^c \frac{x^5}{5!}$$

for some c between 0 and x. Setting $x = -t^2$,

$$e^{-t^2} = 1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} + r_4(-t^2),$$

so that

$$\int_0^1 e^{-t^2} dt = 1 - \frac{1}{3.1!} + \frac{1}{5.2!} - \frac{1}{7.3!} + \frac{1}{9.4!} + \int_0^1 r_4(-t^2) dt$$

$$= 0.74748 \cdots + \int_0^1 r_4(-t^2) dt.$$

Now $r_4(-t^2) = e^c \frac{(-t^2)^5}{5!}$, for some $c \in (-t^2, 0)$. Thus $-\frac{t^{10}}{5!} < r_4(-t^2) < 0$ and hence

$$-\frac{1}{11.5!} < \int_0^1 r_4(-t^2) dt < 0$$

$$-0.00076 < \int_0^1 r_4(-t^2) dt < 0$$

which gives

$$0.7467 < \int_0^1 e^{-t^2} dt < 0.7475.$$

EXAMPLE 10.1.5: The Taylor polynomial of degree n about $a = 0$ for the function $(1+x)^\alpha$ is

$$P_n(x) = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n.$$

If $\alpha = n$, Taylor's Theorem gives the Binomial Theorem as a special case (see Example 10.1.2).

PROOF I OF THEOREM 10.1.1(a): Let C be such that $r_n(x) = C(x-a)^{n+1}$. Thus

$$f(x) - f(a) - \frac{f'(a)}{1!}(x-a) - \frac{f''(a)}{2!}(x-a)^2$$
$$- \cdots - \frac{f^{(n)}(a)}{n!}(x-a)^n - C(x-a)^{n+1} = 0.$$

Consider the function

$$\varphi(t) = f(t) - f(a) - \frac{f'(a)}{1!(t-a)} - \frac{f''(a)}{2!}(t-a)^2$$
$$- \cdots - \frac{f^{(n)}(a)}{n!}(t-a)^n - C(t-a)^{n+1}.$$

Now $\varphi(a) = \varphi'(a) = \varphi^{(2)}(a) = \cdots = \varphi^{(n)}(a) = 0$ and also $\varphi(x) = 0$. Now Rolle's Theorem applied to φ implies $\varphi'(c_1) = 0$ for some c_1 between a and x since $\varphi(a) = \varphi(x) = 0$. Next Rolle's Theorem applied to φ', since $\varphi'(a) = \varphi'(c_1) = 0$, implies $\varphi^{(2)}(c_2) = 0$ for some c_2 between a and x. We may apply Rolle's Theorem successively to $\varphi, \varphi', \ldots, \varphi^{(n)}$ so that finally $\varphi^{(n+1)}(c) = 0$ for some c between a and x. But $\varphi^{(n+1)}(t) = f^{(n+1)}(t) - (n+1)!C$. Thus

$$0 = \varphi^{(n+1)}(c) = f^{(n+1)}(c) - (n+1)!C$$

so that $\quad C = \frac{f^{(n+1)}(c)}{(n+1)!}$. Therefore

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} \, (x-a)^{n+1}.$$

PROOF II OF THEOREM 10.1.1(a): Let $\quad C \quad$ be defined as in the first proof and consider

$$\psi(t) = f(x) - f(t) - \frac{f'(t)}{1!} \, (x-t) - \frac{f''(t)}{2!} \, (x-t)^2$$
$$- \cdots - \frac{f^{(n-1)}}{(n-1)!} \, (x-t)^{n-1} - \frac{f^{(n)}(t)}{n!} \, (x-t)^n - C(x-t)^{n+1}.$$

Now $\quad \psi(a) = \psi(x) = 0 \quad$ and therefore $\quad \psi'(c) = 0 \quad$ for some $\quad c$ between $\quad a \quad$ and $\quad x, \quad$ by Rolle's Theorem. Now

$$\psi'(t) = -f'(t) + [f'(t) - \frac{f^{(2)}(t)}{1!} \, (x-t)]$$

$$+ [\frac{f^{(2)}(t)}{1!} \, (x-t) - \frac{f^{(3)}(t)}{2!} \, (x-t)^2]$$

$$\cdots \cdots \cdots$$

$$+ [\frac{f^{(n-1)}(t)}{(n-2)!} \, (x-t)^{n-2} - \frac{f^{(n)}(t)}{(n-1)!} \, (x-t)^{n-1}]$$

$$+ [\frac{f^{(n)}(t)}{(n-1)!} \, (x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!} \, (x-t)^n]$$

$$+ (n+1)C(x-t)^n$$

$$= - \frac{f^{(n+1)}(t)}{n!} \, (x-t)^n + (n+1)C(x-t)^n$$

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so that $\psi'(c) = 0$ implies $C = \frac{f^{(n+1)}(c)}{(n+1)!}$ as before.

This proof may be modified to obtain the Schlmilch form of the remainder by defining C by $r_n(x) = C(x-a)^m$.

PROOF OF THEOREM 10.1.1(b): The integral form of the remainder is obtained from the Fundamental Theorem of Calculus and successive integrations by parts as follows. First, from the Fundamental Theorem,

$$f(x) = f(a) + \int_a^x f'(t)dt$$

so that $f(x) = f(a) + r_0(x)$ where

$$r_0(x) = \frac{1}{0!} \int_a^x f'(t)dt = -\frac{1}{1!} \int_a^x f'(t)d(x-t)$$

$$= -\frac{1}{1!}f'(t)(x-t)|_{t=a}^{t=x} + \frac{1}{1!} \int_a^x f^{(2)}(t)(x-t)dt$$

$$= \frac{1}{1!}f'(a)(x-a) + \frac{1}{1!} \int_a^x f^{(2)}(t)(x-t)dt.$$

Therefore

$$f(x) = f(a) + \frac{1}{1!}f'(a)(x-a) + r_1(x)$$

and

$$r_1(x) = \frac{1}{1!} \int_a^x f^{(2)}(t)(x-t)dt = -\frac{1}{2!} \int_a^x f^{(2)}(t)d(x-t)^2$$

$$= -\frac{1}{2!}f^{(2)}(t)(x-t)^2|_{t=a}^{t=x} + \frac{1}{2!} \int_a^x f^{(3)}(t)(x-t)^2dt.$$

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Thus

$$f(x) = f(a) + \frac{1}{1!}f'(a)(x-a) + \frac{1}{2!}f^{(2)}(a)(x-a)^2 + r_2(x)$$

where

$$r_2(x) = \frac{1}{2!}\int_a^x f^{(3)}(t)(x-t)^2\,dt.$$

This may also be written in the form

$$r_2(x) = -\frac{1}{3!}\int_a^x f^{(3)}(t)\,d(x-t)^3$$

as before. Proceeding in this way we find

$$f(x) = f(a) + \frac{1}{1!}f'(a)(x-a) + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + r_n(x),$$

where

$$r_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt.$$

$\square$

In practise, we do not need to know $f^{(n+1)}$ precisely to apply Taylor's Theorem. Much information about the remainder may be obtained if we know the sign of $f^{(n+1)}$ or some other bound on $f^{(n+1)}$.

COROLLARY 10.1.6. *Suppose* $|f^{(n+1)}(t)| \le M_{n+1}$ *for all* $t \in [a, x]$ *when* $x > a$ *(or all* $t \in [x, a]$ *when* $x < a$*). Then*

$$f(x) = f(a) + \frac{1}{1!}f'(a)(x-a) + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + r_n(x),$$

where

$$|r_n(x)| \leq \frac{M_{n+1}}{(n+1)!}\, |x-a|^{n+1}.$$

Instead of requiring that $f(x)$ and $p_n(x)$ agree up through the k-th derivative at $x=a$, we might have asked that $f(x)$ and $p_k(x)$ take the same values at distinct points $a_1, \ldots, a_n$ or that $f(x)$ and $p_n(x)$

agree together with some of their derivatives at some distinct points. The same approach may be used to obtain estimates on
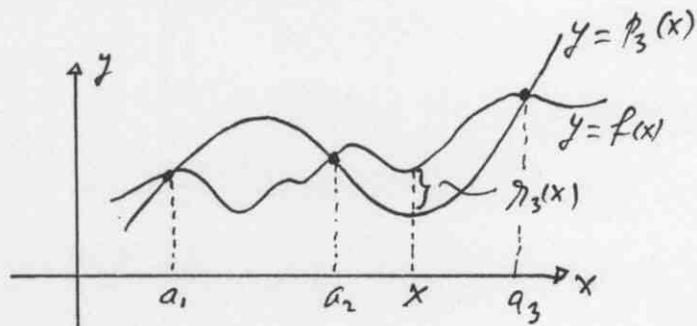
$$r_n(x) = f(x) - p_n(x)$$

in this procedure, which is called *polynomial interpolation*. A good discussion of polynomial interpolation may be found in the book 'Calculus with Analytic Geometry' by Flanders and Price.

In fact, we will obtain some of these results in special cases when we derive error estimates for approximations to integrals.

§10.2. Numerical Approximation of Integrals.

First recall that $P = \{x_0, x_1, \ldots, x_n\}$ is a *partition* of the closed interval $[a,b]$ if $a = x_0 < x_1 < \cdots < x_n = b$. If f is a

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bounded real-valued function on $[a, b]$, then

$$L(P, f) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}),$$

$$U(P, f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$
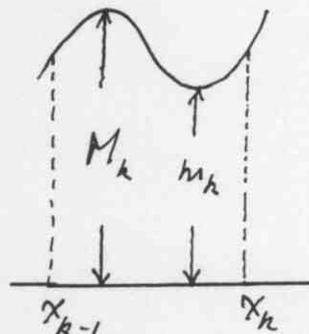
are the lower and upper sums of

f with respect to the partition P.

Any expression of the form

$$S(P, f) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-2}), \quad \begin{array}{c} c_k \in [x_{k-1}, x_k] \\ k = 1, \ldots, n \end{array}$$

is a Riemannian Sum of f with respect to the partition P and

$$L(P, f) \le S(P, f) \le U(P, f)$$

for all such sums. Recall also the definition of the integral: '$\int_a^b f$ exists' means that there is a unique number α such that $L(P, f) \le \alpha \le U(P, f)$ for all partitions P of $[a, b]$ and then we say $\alpha = \int_a^b f$. Equivalently $\alpha = \int_a^b f$ means that, for each $\varepsilon > 0$ there is a partition P_ε of $[a, b]$ such that $P \supset P_\varepsilon \implies |S(P, f) - \alpha| < \varepsilon$ for all Riemann sums $S(P, f)$ corresponding to P.

Apart from a few examples where we found the integral directly from the Riemann sums, our main technique until now for calculating integrals has been by antidifferentiation using the Fundamental Theorem. In cases where it is difficult or impossible to find an antiderivative for f in elementary terms, we still would like to approximate $\int_a^b f$ efficiently. However Riemann sums are not in general a good way to do this.

EXAMPLE 10.2.1: Consider

$$\int_0^1 x^2 dx = \frac{1}{3}x^3 |_0^1 = \frac{1}{3} = 0.\dot{3}.$$

Let $P = \{0, \frac{1}{2}, 1\}$; then

$L(P, f) = 0 \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = 0.125$

$U(P, f) = \frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0.625$

and both are disappointingly

far from the right answers.
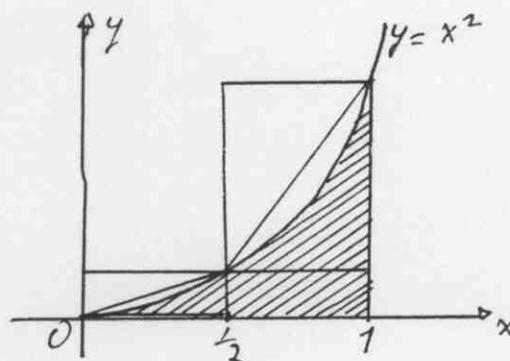
However, if we consider the average of the two numbers (the total area of the trapezoidal regions in the diagram), then we find

$$\frac{1}{2}[L(P, f) + U(P, f)] = \frac{1}{2}[0.125 + 0.625] = \frac{1}{2}(0.75) = 0.375,$$

which is not too bad at all. □

We are now in a position to formulate the *Trapezoidal Rule* for the approximation of $\int_a^b f$.
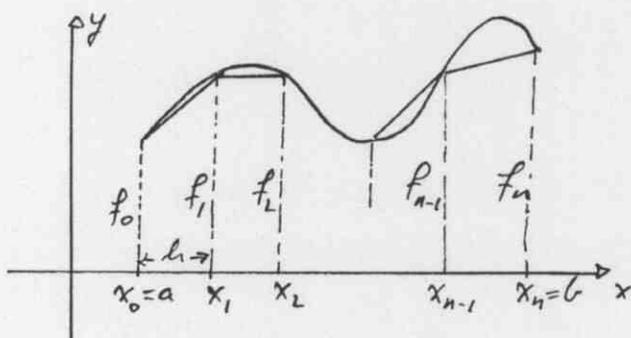
$$h = \frac{b-a}{n}$$

$$x_i = a + i \frac{(b-a)}{n}$$

$$i = 0, 1, \ldots, n$$

$$f_i = f(x_i)$$

$$S_1(P, f) = h[f_0 + f_1 + \cdots + f_{n-1}]$$

$$S_2(P, f) = h[f_1 + f_2 + \cdots + f_n].$$

These are Riemann sums but not necessarily upper or lower sums. Define the Trapezoidal Approximation T_n by $T_n = \frac{1}{2}(S_1 + S_2)$

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so that

$$T_n = \frac{h}{2}\left[f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n\right].$$

EXAMPLE 10.2.2:

$$\int_0^1 x^2\,dx = 0.\dot{3}.$$

$T_2 = \frac{1}{4}\left[0^2 + 2(\frac{1}{2})^2 + 1^2\right] = 0.375$

$T_4 = \frac{1}{8}\left[0^2 + 2(\frac{1}{4})^2 + 2(\frac{1}{2})^2 + 2(\frac{3}{4})^2 + 1^2\right] = 0.34375$

We need to be able to give an estimate on the error

$E_n = \int_a^b f - T_n$ in the Trapezoidal Approximation.
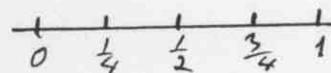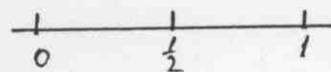
THEOREM 10.2.3 (ERROR ESTIMATE FOR THE TRAPEZOIDAL RULE).

Suppose (a) $|f''(x)| \le M_2, \quad a \le x \le b$,

  (b) $h = \frac{b-a}{n}$,

  (c) $E_n = \int_a^b f - T_n$.

   Then $|E_n| \le \frac{M_2 n h^3}{12} = \frac{M_2(b-a)^3}{12n^2}$.

In Example 10.2.2, the theorem predicts

$$|E_2| \le \frac{2.1}{12.4} = \frac{1}{24} = 0.041\dot{6}$$

$$|E_4| \le \frac{2.1}{12.16} = \frac{1}{96} = 0.01041\dot{6}, \quad \text{since} \quad M_2 = 2.$$

The actual values of the errors are in fact

$$E_2 = 0.041\dot{6}$$

$$E_4 = 0.01041\dot{6}$$

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exactly as predicted. However we cannot in general expect this type of accuracy for the error estimate from Theorem 10.2.3.

EXAMPLE 10.2.4: How large should we take n in the Trapezoidal Rule to approximate $\int_2^3 \frac{1}{x}\, dx = \log(\frac{3}{2})$ so that the error is less than 10^{-5}? Here $f(x) = \frac{1}{x}$, $f''(x) = \frac{-2}{x^3}$ so that

$$|f''(x)| \le \frac{1}{4} = M_2 \quad \text{if} \quad 2 \le x \le 3.$$

We require

$$\frac{M_2(b-a)^3}{12n^2} = \frac{\frac{1}{4}(3-2)^3}{12n^2} = \frac{1}{48n^2} < 10^{-5}$$

or $n^2 > \frac{1}{48}(10)^5$ which holds if $n \ge 46$.

LEMMA 10.2.5.. Suppose (a) $F''(x)$ exists, $a \le x \le b$, and (b) $F(a) = F(b) = 0$. Then, for each $x \in [a, b]$ there is a point $c \in (a, b)$ such that

$$F(x) = \frac{F''(c)}{2!}(x-a)(x-b) \quad (c = c(x)).$$

PROOF: The result is trivial if $x = a$ or $x = b$ so we assume $x \in (a, b)$. Choose C so that

$$F(x) = C(x-a)(x-b)$$

and consider

$$\varphi(t) = F(t) - C(t-a)(t-b).$$

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Now $\varphi(a) = \varphi(x) = \varphi(b)$

so $\varphi'(x_1) = \varphi'(x_2) = 0$
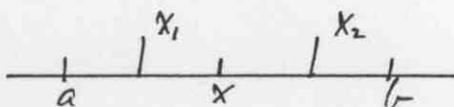
for some $x_1 \in (a, x)$ and $x_2 \in (x, b)$, by Rolle's Theorem.

This in turn implies, by Rolle's Theorem,

$$\varphi''(c) = 0, \quad \text{for some} \quad c \in (x_1, x_2).$$

Thus

$$0 = \varphi''(c) = F''(c) - 2!C$$

so that

$$C = \frac{F''(c)}{2!} .$$

$\square$

LEMMA 10.2.6. *Suppose (a)* $|f''(x)| \leq M_2, \quad 0 \leq x \leq h,$

(b) $t(x) = Ax + B$ *satisfies* $f(0) = t(0), \ f(h) = t(h).$

Then

$$|\int_0^h [f(x) - t(x)]dx| \leq \frac{M_2 h^3}{12} .$$

PROOF: Let $F(x) = f(x) - t(x)$;

$F(0) = F(h) = 0.$ If $x \in [0, h]$,

then the preceding lemma

shows that, for some $c \in (0, h)$,

$$F(x) = \frac{F''(c)}{2!} \ x(x - h)$$
$$= \frac{f''(c)}{2!} \ x(x - h)$$

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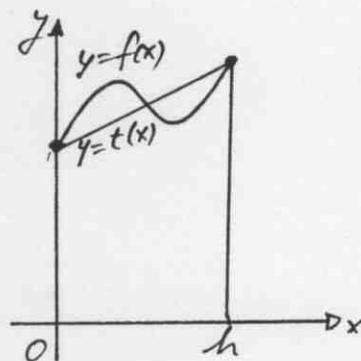
and therefore

$$|F(x)| \le \frac{M_2}{2!} \, x(h - x), \quad 0 \le x \le h$$

or equivalently

$$-\frac{M_2}{2!} \, x(h - x) \le F(x) \le \frac{M_2}{2!} \, x(h - x).$$

Now $\int_0^h x(h - x)dx = (\frac{x^2}{2} h - \frac{x^3}{3})|_0^h = h^3(\frac{1}{2} - \frac{1}{3}) = \frac{h^3}{6}$, so that $-\frac{M_2}{2} \frac{h^3}{6} \le \int_0^h F(x)dx \le \frac{M_2}{2} \cdot \frac{h^3}{6}$ and therefore

$$|\int_0^h [f(x) - t(x)]dx| \le \frac{M_2 h^3}{12} \, .$$

$\square$

If we apply the result of Lemma 10.2.6 to each of the n partition intervals in Theorem 10.2.3, we find

$$|E_n| = |\int_a^b f(x)dx - T_n| \le \frac{M_2 n h^3}{12} = \frac{M_2(b - a)^3}{12n^2} \, .$$

$\square$

Simpson's Rule replaces the straight line approximation of the Trapezoidal Rule by a parabolic approximation. A parabola $y = Ax^2 + Bx + C$ is uniquely determined by 3 points on the curve. We now take $h = \frac{b-a}{2n}$ and partition $[a, b]$ into $2n$ subintervals of length h

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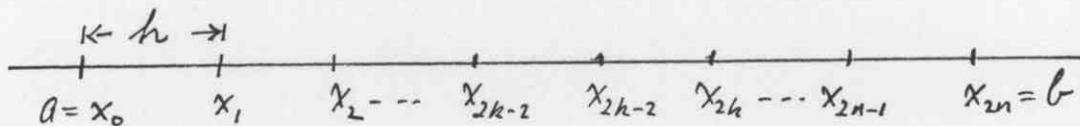
On each interval $[x_{2k-2}, x_{2k}]$,

we define $s(x) = Ax^2 + Bx + C$

so that

$s(x_i) = f(x_i), \quad i = 2k-2, 2k-1, 2k.$

To discuss the approximation,

we locate the origin at x_{2k-1}

so that

$x_{2k-2} = -h,\ x_{2k-1} = 0,\ x_{2k} = h.$

Then
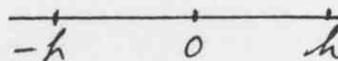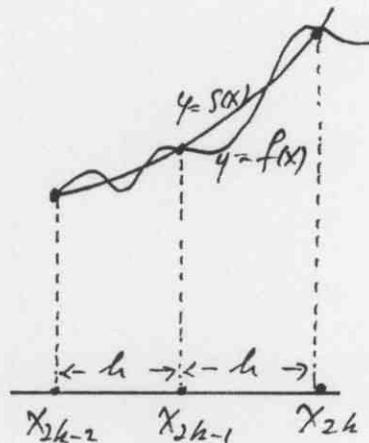
$$S = \int_{-h}^{h} s(x)\,dx = \int_{-h}^{h} (Ax^2 + Bx + C)\,dx$$

$$= A\,\frac{x^3}{3} + B\,\frac{x^2}{2} + Cx\Big|_{-h}^{h} = 2A\,\frac{h^3}{3} + 2Ch$$

$f(0) = s(0) = C$

$$\left.\begin{array}{l} f(-h) = s(-h) = Ah^2 - Bh + C \\ f(h) = s(h) = Ah^2 + Bh + C \end{array}\right\} \implies f(-h) + f(h) = 2Ah^2 + 2C.$$

Therefore $\quad C = f(0), \quad A = \frac{1}{2h^2}\left[f(-h) + f(h)\right] - \frac{1}{h^2} f(0)\quad$ and

$$S = \frac{1}{3}\left[f(-h) + f(h)\right]h - \frac{2}{3} f(0)h + 2f(0)h$$

$$= \frac{h}{3}\left[f(-h) + 4f(0) + f(h)\right].$$

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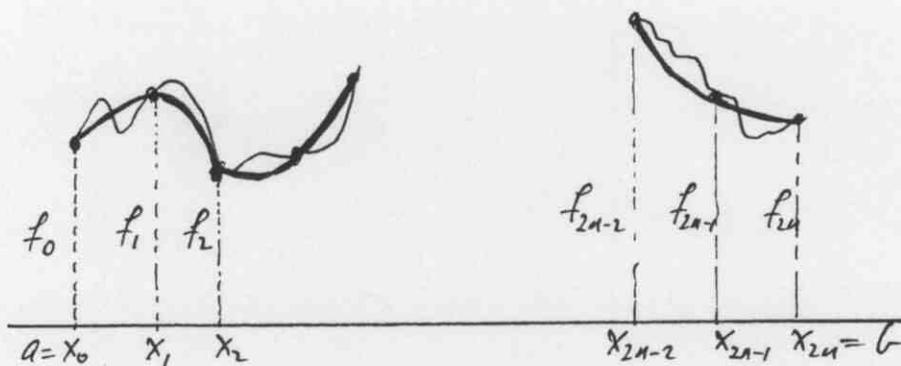
$$h = \frac{b-a}{2n}$$

$$x_i = a + \frac{i(b-a)}{2n}$$

$$i = 0, 1, \cdots, 2n$$

$$f_i = f(x_i)$$

Dividing the interval $[a,b]$ into $2n$ subintervals of length $h = \frac{b-a}{2n}$ we define the Simpson approximation

$$S_n = \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 2f_{2n-2} + 4f_{2n-1} + f_{2n}\right].$$

THEOREM 10.2.7 (ERROR ESTIMATE FOR SIMPSON'S RULE).

Suppose (a) $|f^{(4)}(x)| \le M_4, \quad a \le x \le b$

(b) $h = \frac{b-a}{2n}$

(c) $E_n = \int_a^b f - S_n.$

Then

$$|E_n| \le \frac{M_4 n h^5}{90} = \frac{M_4(b-a)^5}{2,880 n^4}.$$

LEMMA 10.2.8. Suppose

(a) $F^{(4)}(x)$ exists, $a \le x \le b$, and
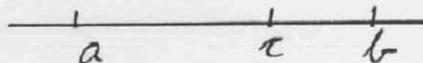
(b) $F(a) = F(c) = F'(c) = F(b) = 0$ for some $c \in (a,b)$.

Then, for each $x \in [a,b]$, there is a point $p \in (a,b)$ such that

$$F(x) = \frac{F^{(4)}(p)}{4!}(x-a)(x-c)^2(x-b) \quad (p = p(x)).$$

PROOF: The statement is true for all $p \in (a,b)$ if $x = a, b$ or c.

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If $x \neq a, b, c$, choose C so that

$$F(x) = C(x-a)(x-c)^2(x-b)$$

and consider

$$\varphi(t) = F(t) - C(t-a)(t-c)^2(t-b).$$

Now $\varphi(a) = \varphi(c) = \varphi'(c) = \varphi(b) = \varphi(x) = 0$, so several successive applictions of Rolle's Theorem show

$$0 = \varphi^{(4)}(p) = F^{(4)}(p) - 4!C$$

for some $p \in (a, b)$ and therefore

$$C = \frac{F^{(4)}(p)}{4!}.$$

Thus

$$F(x) = \frac{F^{(4)}(p)}{4!}(x-a)(x-c)^2(x-b),$$

as asserted.

$\square$

LEMMA 10.2.9. *Suppose*

(a) $|f^{(4)}(x)| \leq M_4$, $-h \leq x \leq h$, and X

(b) $s(x) = Ax^2 + Bx + C$ *satisfies*

$$f(-h) = s(-h), \quad f(0) = s(0), \quad f(h) = s(h).$$

Then

$$\left| \int_{-h}^{h} [f(x) - s(x)]dx \right| \leq \frac{M_4 h^5}{90}.$$

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PROOF: First observe that

$$\int_{-h}^{h} [f(x) - s(x)]dx = \int_{-h}^{h} [f(x) - s(x) + \{f'(0) - s'(0)\} \frac{x(x^2 - h^2)}{h^2}]dx,$$

since $\int_{-h}^{h} x(x^2 - h^2)ds = 0$ as the integrand is an odd function. Let

$$F(x) = f(x) - s(x) + \{f'(0) - s'(0)\} \frac{x(x^2 - h^2)}{h^2}, \quad -h \leq x \leq h.$$

Then $F^{(4)}(x)$ exists, $-h \leq x \leq h$ and

$$F(-h) = F(0) = F'(0) = F(h)$$

since $D[x(x^2 - h^2)]_{x=0} = -h^2$. If $x \in [-h, h]$, then Lemma 10.2.8 implies that, for some $p \in (-h, h)$,

$$F(x) = \frac{F^{(4)}(p)}{4!} x^2(x^2 - h^2)$$

$$= \frac{f^{(4)}(p)}{4!} x^2(x^2 - h^2)$$

and therefore, for all $x \in [-h, h]$,

$$|F(x)| \leq \frac{M_4}{4!} x^2(h^2 - x^2)$$

or equivalently

$$-\frac{M_4}{4!} x^2(h^2 - x^2) \leq F(x) \leq \frac{M_4}{4!} x^2(h^2 - x^2).$$

Now

$$\int_{-h}^{h} x^2(h^2 - x^2)dx = [\frac{x^3 h^2}{3} - \frac{x^5}{5}]_{-h}^{h} = \frac{4h^5}{15}$$

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so that

$$-\frac{M_4}{4!} \cdot \frac{4h^5}{15} \leq \int_{-h}^{h} F(x)dx \leq \frac{M_4}{4!} \cdot \frac{4h^5}{15} \,,$$

$$-\frac{M_4}{90} h^5 \leq \int_{-h}^{h} [f(x) - s(x)]dx \leq \frac{M_4}{90} h^5 \,,$$

and therefore $\left| \int_{-h}^{h} [f(x) - s(x)]dx \right| \leq \frac{M_4}{90} h^5.$

$\square$

We may now apply the result of Lemma 10.2.9 to successive pairs of adjacent intervals in Simpson's Rule. Since there are n pairs of such intervals,

$$\left| \int_{a}^{b} f(x)dx - S_n \right| \leq \frac{M_4 n h^5}{90} \,.$$

$\square$

EXAMPLE 10.2.10 We will plan the computation of $\log 2 = \int_{1}^{2} \frac{1}{x} dx$ to five decimal places using both the Trapezoidal and Simpson's rules. This degree of accuracy using upper or lower sums requires a partition of $[1, 2]$ into approximately 10^5 intervals (see Problem 10.15). An approximation scheme will have the required degree of accuracy if the error E_n satisfies $|E_n| < 5 \times 10^{-6}$. Here

$$f(x) = \frac{1}{x} \,, \ f'(x) = \frac{-1}{x^2} \,, \ f^{(2)}(x) = \frac{2}{x^3} \,, \ f^{(3)}(x) = \frac{-6}{x^4} \,, \ f^{(4)}(x) = \frac{24}{x^5}$$

so that, on $[1, 2]$, we may take

$$M_2 = 2, \quad M_4 = 24.$$

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In the Trapezoidal Rule, we have the required degree of accuracy if

$$5 \times 10^{-6} > \frac{M_2(b-a)^3}{12n^2} = \frac{1}{6n^2}, \quad \text{or} \quad n \geq 183.$$

In Simpson's Rule, we will need

$$5 \times 10^{-6} > \frac{M_4(b-a)^5}{2,880n^4} = \frac{1}{120n^4}, \quad \text{or} \quad n \geq 7.$$

Thus we need 184 values of f for the Trapezoidal Rule and 15 values for Simpson's Rule.

EXAMPLE 10.2.11 Estimate the accuracy of the trapezoidal and Simpson approximations to $\int_0^2 \sin(x^2)dx$ if 5 function values are used. Since $0 \leq x \leq 2$ and

$$f(x) = \sin(x^2)$$

$$f'(x) = 2x \, \cos(x^2)$$

$$f^{(2)}(x) = 2 \, \cos(x^2) - 4x^2 \sin(x^2)$$

$$f^{(3)}(x) = -12x \, \sin(x^2) - 8x^3 \cos(x^2)$$

$$f^{(4)}(x) = -12 \, \sin(x^2) - 48x^2 \cos(x^2) + 16x^4 \sin(x^2).$$
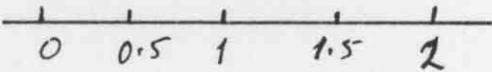
We may take $M_2 = 18$, $M_4 = 460$. Therefore

$$\left| \int_0^2 \sin(x^2)dx - T_n \right| \leq \frac{18.2^3}{12.16} = \frac{3}{4} = 0.75$$

and

$$\left| \int_0^2 \sin(x^2)dx - S_n \right| \leq \frac{460.2^5}{2,880.2^4} = \frac{23}{72} = 0.31\dot{9}\dot{4}.$$

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§10.3. Newton's Method

We conclude this chapter with a discussion of Newton's method for the approximation of roots of equations in the form $f(x) = 0$. The idea behind the method is quite simple: make an initial guess x_0 at the root; find where the tangent at $(x_0, f(x_0))$ to $y = f(x)$ intersects the x-axis to find x_1, the next approximation. Next consider the tangent to $y = f(x)$ at $(x_1, f(x_1))$ to find x_2 and continue iteratively like this. To find the general scheme,
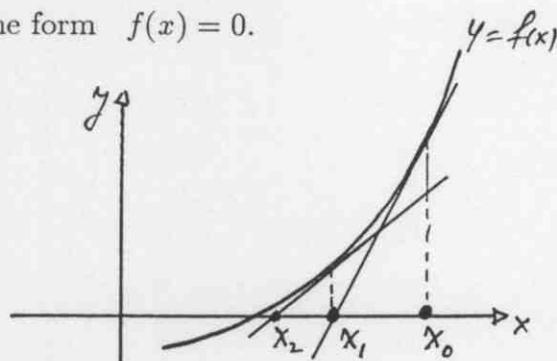
$$y - f(x_n) = f'(x_n)(x - x_n)$$

is the equation of the tangent line to $y = f(x)$ at $(x_n, f(x_n))$. It intersects the x-axis $(y = 0)$ at $(x_{n+1}, 0)$ where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2.$$

Thus the choice of x_0 determines the sequence $\{x_n\}$ as long as each term x_n is in the domain of f, f' and $f'(x_n) \neq 0$.

PROPOSITION 10.3.1. *Suppose*

(a) $\displaystyle\lim_{n \to \infty} x_n = x_*$ *and*

(b) f' *exists and is bounded near* x_*.

   Then $f(x_*) = 0$.

PROOF: This follows from

$$f(x_n) = (x_n - x_{n+1})f'(x_n)$$

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which implies

$$f(x_*) = \lim_{n \to \infty} f(x_n) = 0.$$

□

There is no guarantee that the sequence of iterates is convergent. For example, if

$$f(x) = x^{1/3}, \quad f'(x) = \frac{1}{3} x^{-2/3}$$

and $x - \frac{f(x)}{f'(x)} = x - 3x = -2x$. Thus

$x_{n+1} - x_n - \frac{f(x_n)}{f'(x_n)} = -2x_n$

implies $x_n = (-2)^n x_0$ and

$\{x_n\}$ is divergent if $x_0 \neq 0$.
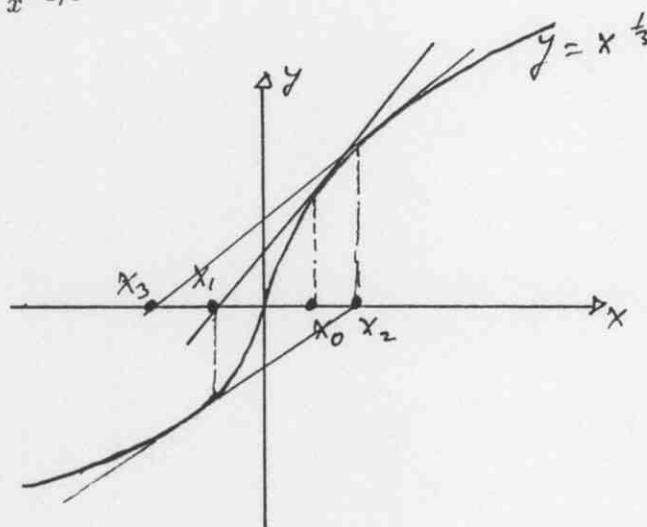
Here the accuracy of our approximation deteriorates with each successive step.

On the other hand the procedure works very well in some cases. For example, if we wish to approximate $\sqrt{5}$, the positive root of $x^2 - 5 = 0$, we apply Newton's method to $f(x) = x^2 - \cancel{\theta} : \quad 5$
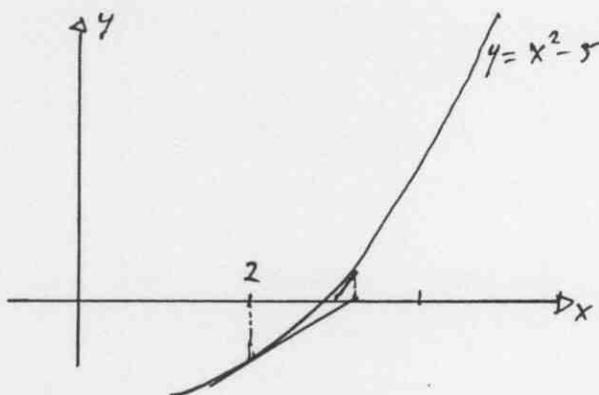
$$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n}$$

$$x_{n+1} = \frac{1}{2}(x_n + \frac{5}{x_n}).$$

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Now, the Intermediate Value Theorem for continuous functions tells us that $2 < \sqrt{5} < 3$. With an initial point $x_0 = 2$, we find

$$x_1 = 2.25$$

$$x_2 = 2.236111111$$

$$x_3 = 2.236067978.$$

Thus we can achieve the accuracy of the calculator in 3 iterations.

We have seen that, if the Newton iterations converge, they converge to a root of $f(x) = 0$. We now need

(a) conditions which guarantee $\{x_n\}$ is convergent

(b) an estimate on the error $x_n - x_*$.

PROPOSITION 10.3.2. *Suppose*

(a) $|f'(x)| \geq m > 0$, $\quad |f''(x)| \leq M$, $\quad a < x < b$

(b) $x_k \in (a, b)$ $\quad$ and $\quad$ $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

(c) $f(x_*) = 0$.

Then

$$|x_{k+1} - x_*| \leq \frac{M}{2m} |x_k - x_*|^2.$$

PROOF:

$$|x_{k+1} - x_*| = |x_k - \frac{f(x_k)}{f'(x_k)} - x_*|$$

$$= \frac{1}{|f'(x_k)|}| - f(x_k) - f'(x_k)(x_* - x_k)|$$

$$= \frac{1}{|f'(x_k)|}|f(x_*) - f(x_k) - f'(x_k)(x_* - x_k)|,$$

$$\text{since} \quad f(x_*) = 0$$

$$= \frac{1}{|f'(x_k)|} \left| \frac{f''(c)}{2!} (x_* - x_k)^2 \right| \quad \text{for some} \quad c$$

$$\text{between} \quad x_k \quad \text{and} \quad x_* \quad \text{(Taylor's Theorem)}$$

$$= \frac{|f''(c)|}{2|f'(x_k)|} |x_k - x_*|^2$$

$$\leq \frac{M}{2m} |x_k - x_*|^2.$$

COROLLARY 10.3.3. *Under the conditions of the preceding proposition, if* $x_0, \ldots, x_{n-1} \in (a, b)$ *then*

$$|x_n - x_*| \leq \left(\frac{M}{2m}\right)^{2^n - 1} |x_0 - x_*|^{2^n},$$

where $x_0, \ldots, x_n$ *are the Newton iterates.*

PROOF: This follows from

$$|x_{k+1} - x_*| \leq \frac{M}{2m} |x_k - x_*|^2, \quad k = 0, 1, 2, \ldots$$

and induction. The assertion is true for $n = 1$. Now, if it holds for $n = k$,

$$|x_k - x_*| \leq \left(\frac{M}{2m}\right)^{2^k - 1} |x_0 - x_*|^{2^k}$$

so that

$$|x_{k+1} - x_*| \leq \frac{M}{2m}|x_k - x_*|^2$$

$$\leq \frac{M}{2m}\left[\left(\frac{M}{2m}\right)^{2^k-1}|x_0 - x_*|^{2^k}\right]^2$$

$$= \left(\frac{M}{2m}\right)^{2^{k+1}-1}|x_0 - x_*|^{2^{k+1}}$$

it also holds for $n = k+1$. The assertion of the corollary holds for all n.

□

Corollary 10.3.3 shows that if x_0 is chosen sufficiently close to x_*, specifically

$$\frac{M}{2m}|x_0 - x_*| < 1$$

then, by the Squeeze Principle, $\lim_{n \to \infty} x_n = x_*$ and moreover it gives an estimate on the difference $x_n - x_*$. The choice of initial point x_0 sufficiently close to x_* may be achieved by some consideration such as the Intermediate Value Theorem for continuous functions.

EXAMPLE 10.3.4 In approximating $\sqrt{5}$ we considered $f(x) = x^2 - 5$. Here

$$f'(x) = 2x, \quad f''(x) = 2.$$

Now $f(2) = -1 < 0 < 4 = f(3)$ so that $x_* = \sqrt{5} \in (2,3)$;

$$|f'(x)| \geq 4 = m \quad \text{and} \quad |f''(x)| = 2 = M$$

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if $x \in [2,3]$. If we choose $x_0 = 2$, then

$$\frac{M}{2m}|x_0 - x_*\| = \frac{2}{8}|2 - x_*| < \frac{2}{8}|2 - 3| = \frac{1}{4}.$$

Thus Corollary 10.3.3 implies

$$|x_n - \sqrt{5}| = |x_n - x_*| \leq \left(\frac{M}{2m}\right)^{2^n - 1}|x_0 - x_1|^{2^n} < 2^{-(2^n - 1)}.$$

In particular $|x_3 - \sqrt{5}| < 2^{-7}$; we have found that the actual value of $|x_3 - \sqrt{5}|$ is much smaller than this. For $n = 5$ we have

$$|x_5 - \sqrt{5}| < 2^{-31} < 5 \times 10^{-10};$$

the theory predicts that 5 iterations gives 4-place accuracy which we actually achieved in 3 iterations. To achieve 20-place accuracy we require

$$|x_n - \sqrt{5}| < 5 \times 10^{-21}$$

which will hold if $2^{-N} \leq 5 \times 10^{-21}$, $N = 2^n - 1$ or equivalently $2^N \geq 0.2 \times 10^{21}$. Now $2^{68} = (0.29\ldots) \times 10^{21}$ so we require $2^n - 1 = N \geq 68$ which is achieved for $n = 7$. Thus 7 iterations give at least 20-place accuracy.

Problems

10.1. Show that $\sin(a + h)$ differs from $\sin a + h \cos a$ by not more than $\frac{1}{2} h^2$.

HINT: $f(x) = f(a) + f'(a)(x - a) + r_1(x)$, $x = a + h$, $f(x) = \sin x$.

10.2. Find a bound of the form Cx^3 for each of the following approximations if $x \geq 0$

(a) $(1 + x)^{1/2} \simeq 1 + \frac{1}{2} x - \frac{1}{8} x^2$

(b) $(1 + x)^{1/3} \simeq 1 + \frac{1}{3} x - \frac{1}{9} x^2$.

10.3. Express the polynomial $1 - 2x + x^2 - 3x^5$ in powers of $x - 1$ by using Taylor's Theorem.

10.4. On one set of coordinate axes, sketch the graphs of $\sin x$ and its Taylor polynomials x, $x - \frac{x^3}{3!}$, $x - \frac{x^3}{3!} + \frac{x^5}{5!}$, paying careful attention to the points where the curves intersect the x-axis.

10.5. For each of the functions, find the 5-th degree Taylor polynomial about $x = a$ and give the Lagrange form of the remainder

(a) $\sin x$, $a = \frac{\pi}{4}$

(b) x^{-3}, $a = -1$.

10.6. (a) Show that the functions $\sin x$, $\cos x$ have Taylor polynomials

$$S_{2n-1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$$

$$C_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

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(b) Show that for all $x \in \mathbb{R}$

$$|\sin x - S_{2n-1}(x)| \le \frac{|x|^{2n+1}}{(2n+1)!}, \quad |\cos x - C_{2n}(x)| \le \frac{x^{2n+2}}{(2n+2)!}$$

10.7. Prove that $\lim\limits_{n \to \infty} (1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1}\frac{1}{n}) = \log 2$.

HINT: Apply Taylor's Theorem to $\log(1 + x)$ about $x = 0$.

Show $\lim\limits_{n \to \infty} r_n(1) = 0$.

10.8. Estimate the error in the approximation

$$\int_0^1 \frac{1}{(1 + x^{20})^{1/4}} \, dx \simeq \int_0^1 (1 - \frac{x^{20}}{4}) dx.$$

10.9. Consider $\int_1^3 \sin x \, dx$.

(a) If $[1, 3]$ is partitioned into 10 subintervals of length $h = 0.2$, show that the Trapezoidal $(n = 10)$ and Simpson's $(n = 5)$ approximations have errors less than 0.0067 and 0.000018 respectively.

(b) If $h = 0.1$, show that the errors are less than 0.0017 and 0.0000012 respectively.

10.10. (a) Show $\int_0^1 \frac{1}{1+x^2} \, dx = \frac{\pi}{4}$.

(b) Give the Trapezoidal estimation for the integral in (a) with $h = 0.25$.

(c) Give the Simpson's estimation for the integral in (a) with $h = 0.25$.

(d) In each of (b), (c), compare the actual error with the error bounds given by the theorems.

10.11. Into how many equal intervals should the interval $[1, 4]$ be

partitioned to guarantee an error less than 10^{-5} in the approximation of $\int_1^4 \frac{1}{x^2} dx$ using Simpson's rule? You should keep the number as small as you can.

10.12. For what functions f does the Trapezoidal Rule $(n = 1)$ give the exact value? Simpson's Rule $(n = 1)$? Why?

10.13. Show that the closest integer to $\frac{n}{e}$ is divisible by $n - 1$. [What was Taylor's first name?]

10.14. Suppose f is twice differentiable on $[0, 1]$ and $f(0) = 0$, $f(1) = 1$, $f'(0) = 0$, $f'(1) = 0$. Show that $|f''(x)| \geq 4$ for some $x \in (0, 1)$.

10.15. Let $h = \frac{b-a}{n}$, $f_i = f(a + \frac{i}{n}(b-a))$ and

$$L_n = h[f_0 + f_1 + \cdots + f_{n-1}],$$
$$R_n = h[f_1 + f_2 + \cdots + f_n],$$
$$S_n = h[f_{1/2} + f_{3/2} + \cdots + f_{n-\frac{1}{2}}].$$

L_n, R_n, S_n are called the *left endpoint, right endpoint* and *midpoint* approximations of $\int_a^b f(x)dx$ respectively.
(a) If $|f'(x)| \leq M_1$, $a \leq x \leq b$, show that

$$\left| \int_a^b f(x)dx - L_n \right| \leq \frac{M_1 nh^2}{2} = \frac{M_1(b-a)^2}{2n}.$$

This error estimate is also valid for the right endpoint approximation.
(b) If $|f''(x)| \leq M_2$, $a \leq x \leq b$, show that

$$\left| \int_a^b f(x)dx - S_n \right| \leq \frac{M_2 nh^3}{24} = \frac{M_2(b-a)^3}{24n^2}.$$

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10.16. Suppose

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Show that

$$\lim_{x \to a} \frac{f(x) - p_n(x)}{(x-a)^n} = 0.$$

[What was l'Hospital's first name?]

10.17. Suppose that $n \geq 2$ and

$$f'(a) = \ldots\ldots\cdots = f^{(n-1)}(a) = 0,\ f^{(n)}(a) \neq 0.$$

(a) If n is even and $f^{(n)}(a) > 0$, then f has a local minimum at a.

(b) If n is even and $f^{(n)}(a) < 0$, then f has a local maximum at a.

(c) If n is odd, then f has neither a local maximum nor minimum at a.

Consider #10.16. This problem has the dreaded second derivative test as a special case.]

10.18. We wish to determine the volume of water per minute which flows past a line across the river. If we are provided with an instrument which measures depth and an instrument which measures the current speed at any depth, plan the solution of the problem making reference to any assumptions made.

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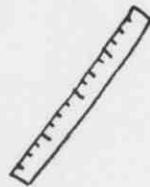
10.19. Show that $x^3 + x - 1 = 0$ has exactly one real root and approximate it to 4 decimals.

10.20. Plan the location of x_0 and the number of Newton iterates for the solution of $x + \log x = 0$ so that an error less than 10^{-8} is achieved.

10.21. Suppose (a) $|F^{(3)}(x)| \le M_3$, $a \le x \le b$.

(b) $F(p) = F(q) = F(r) = 0$, $p, q, r \in [a.b]$ $p < q < r$.

Show that, if $a \le x \le b$,

$$|F(x)| \le \frac{M_3}{3!} |(x - p)(x - q)(x - r)|.$$

10.22. How large should N be in order that

$$\left| \int_0^1 e^{x^2} dx - \sum_{n=0}^{N} \frac{1}{(2n + 1)n!} \right| < 0.0001?$$

10.23. Find $\int_0^1 \frac{\sin x}{x} dx$ correct to two places of decimals.

10.24. Suppose f is differentiable in $(0, \infty)$ and $\lim_{x \to \infty} f(x)$ exists. Show that $\lim_{x \to \infty} f'(x)$ might not exist but that, if it does exist, it *must* be 0.

10.25. Suppose $\lim_{x \to \infty} f(x), \lim_{x \to \infty} f''(x)$ both exist. Show that $\lim_{x \to \infty} f'(x)$ also exists and that

$$\lim_{x \to \infty} f'(x) = 0, \quad \lim_{x \to \infty} f''(x) = 0.$$

XI. INFINITE SERIES AND IMPROPER INTEGRALS

§11.1. Sequences.

We review some of the properties of sequences from Chapter II,

(a) $\lim_{n \to \infty} a_n = \ell$ means: If $\varepsilon > 0$ there exists N such that

$$n \geq N \Longrightarrow |a_n - \ell| < \varepsilon.$$

We say: ' $\{a_n\}$ is *convergent* with limit ℓ '. A sequence which is not convergent is said to be *divergent*.

(b) The sequence $\{a_n\}$ is convergent if and only if it is a *Cauchy sequence:* If $\varepsilon > 0$ there exists N such that

$$m, n \geq N \Longrightarrow |a_n - a_m| < \varepsilon.$$

(c) A sequence is convergent if and only if every subsequence is convergent. Notice that the limit was not mentioned – is it still OK?

(d) A convergent sequence is bounded but a bounded sequence need not be convergent.

(e) A bounded sequence has a convergent subsequence. This is the Bolzano-Weierstrass Theorem.

(f) A monotone (increasing or decreasing) sequence is convergent if and only if it is bounded.

(g) Suppose $\lim_{n \to \infty} a_n = \ell$, $\lim_{n \to \infty} b_n = m$. Then

    (i) $\lim_{n \to \infty}(a_n + b_n) = \ell + m$,

    (ii) $\lim_{n \to \infty} a_n b_n = \ell m$,

    (iii) $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\ell}{m}$, if $m \neq 0$.

(h) Recall the important *Squeeze Principle*: If $a_n \leq c_n \leq b_n$, $n = 1, 2, \ldots$ and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \ell$, then

$$\lim_{n \to \infty} c_n = \ell.$$

(i) Recall the connection with the limits of more general functions f: Suppose there are points x in the domain of f arbitrarily close to but distinct from a. Then $\lim_{x \to a} f(x) = \ell \iff$ each sequence $\{x_n\}$ in the domain of f such that

$$x_n \neq a, \quad \lim_{n \to \infty} = a$$

satisfies

$$\lim_{n \to \infty} f(x_n) = \ell.$$

(j) Some important special limits are:

   (i) $\lim_{n \to \infty} \frac{1}{n^p} = 0$, if $p > 0$

   (ii) $\lim_{n \to \infty} c^n = 0$, if $|c| < 1$

   (iii) $\lim_{n \to \infty} c^{1/n} = 1$, if $c > 0$

   (iv) $\lim_{n \to \infty} n^{1/n} = 1$

   (v) $\lim_{n \to \infty} \frac{\log n}{n^p} = 0$ if $p > 0$

   (vi) $\lim_{n \to \infty} \frac{n^p}{e^n} = 0$, if $p \in \mathbb{R}$

   (vii) $\lim_{n \to \infty} (1 + \frac{x}{n})^n = e^x$, if $x \in \mathbb{R}$.

It would be a good idea to think your way through proofs of some of these.

§11.2. Infinite Series.

Let $\{a_n\}$ be a sequence of real numbers. We will write

$$\sum_{k=1}^{\infty} a_k = s$$

if $\lim_{n \to \infty} s_n = s$, where $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$. We will say 'the *series* $\sum_{k=1}^{\infty} a_k$ is *convergent* with sum s'. Thus convergence of the series $\sum_{k=1}^{\infty} a_k$ is equivalent to the convergence of the sequence of *partial sums* $\{s_n\}$,

We will sometimes write

$$a_1 + a_2 + a_3 + \cdots = s.$$

The fact that the first term in the series is a_1 is not important and, for example,

$$\sum_{k=0}^{\infty} a_k = a_0 + \sum_{k=1}^{\infty} a_k = a_0 + a_1 + a_2 + \sum_{k=3}^{\infty} a_k$$

with each of the infinite series being convergent if and only if one of them is convergent.

EXAMPLE 11.2.1. We are accustomed to writing

$$\frac{1}{3} = 0.333\ldots.$$

This is the same as $\frac{1}{3} = \lim_{n \to \infty}(\frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n})$ or $\frac{1}{3} =$

$\sum_{k=1}^{\infty} \frac{3}{10^k}$. To see this, consider

$$
\begin{aligned}
s_n &= \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n} \\
&= \frac{3}{10} \left(1 + \frac{1}{10} + \cdots + \frac{1}{10^{n-1}} \right) \\
&= \frac{3}{10} \frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}} \xrightarrow[n \to \infty]{} \frac{3}{10} \frac{1}{1 - \frac{1}{10}} \quad \text{since} \quad \frac{1}{10^n} \xrightarrow[n \to \infty]{} 0 \\
&= \frac{3}{10} \cdot \frac{10}{9} = \frac{1}{3} \; .
\end{aligned}
$$

More generally

$$
\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k, \quad \text{if} \quad |x| < 1 \quad \text{(The Geometric Series)}
$$

and this series is divergent if $|x| \geq 1$. To see this, consider

$$
s_n = \sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}, \quad x \neq 1.
$$

Therefore $\lim_{n \to \infty} s_n = \frac{1}{1-x}$, if $|x| < 1$, and s_n is divergent if $|x| > 1$ or $x = -1$. If $x = 1$, $s_n = n + 1$, an unbounded and therefore divergent sequence.

EXAMPLE 11.2.2. $\sum_{k=0}^{\infty} (-1)^k$ is divergent since the partial sums are alternately 1 and 0.

EXAMPLE 11.2.3.

$$
\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1,
$$

since $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, so that

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)}$$

$$= [(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{n-1} - \frac{1}{n}) + (\frac{1}{n} - \frac{1}{n+1})]$$

$$= \frac{1}{1} - \frac{1}{n+1} \quad \text{(the sum telescopes)}$$

$$= 1 - \frac{1}{n+1} \xrightarrow[n \to \infty]{} 1.$$

$\square$

If $\sum_{k=1}^{\infty} a_k$ is convergent, then its sum $s = \lim_{n \to \infty} s_n$. There-fore $\lim_{n \to \infty} s_{n-1} = s$ and, since $s_n - s_{n-1} = a_n$,

$$\lim_{n \to \infty} (s_n - s_{n-1}) = s - s = 0$$

$$\implies \lim_{n \to \infty} a_n = 0.$$

We thus have the following *necessary* condition for convergence of an infinite series

PROPOSITION 11.2.4. *If* $\sum_{k=1}^{\infty} a_k$ *is convergent, then*

$$\lim_{n \to \infty} a_n = 0.$$

The converse is not true i.e. $\lim_{n \to \infty} a_n = 0 \nRightarrow \sum_{k=1}^{\infty} a_k$ *is conver-gent. For example,* $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ *is divergent since*

$$s_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \frac{n}{\sqrt{n}} = \sqrt{n}$$

so that $\{s_n\}$ *is unbounded and therefore divergent.*

The Cauchy Criterion (Theorem 2.5.2) for convergence of a sequence translates to the Cauchy Criterion for convergence of an infinite series.

THEOREM 11.2.5 (CAUCHY CRITERION). $\sum\limits_{k=1}^{\infty} a_k$ is convergent if and only if, for each $\varepsilon > 0$, there exists N such that

$$m, n \geq N \implies |\sum_{k=m}^{n} a_k| < \varepsilon.$$

Notice that the Cauchy criterion implies Proposition 11.2.4.

EXAMPLE 11.2.6. $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ is divergent since

$$\sum_{k=n}^{2n} \frac{1}{k} = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} > \frac{n}{2n} = \frac{1}{2}.$$

Thus, for $\varepsilon = \frac{1}{2}$, we can produce no N for the Cauchy Criterion.

THEOREM 11.2.6. Suppose

$$\sum_{k=1}^{\infty} a_k = s, \quad \sum_{k=1}^{\infty} b_k = t.$$

Then

$$\sum_{k=1}^{\infty} (a_k + b_k) = s + t$$

and

$$\sum_{k=1}^{\infty} c a_k = cs, \quad if \quad c \in \mathbb{R}.$$

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PROOF: This follows easily from the definition

$$s_n = \sum_{k=1}^{n} a_n, \quad t_n = \sum_{k=1}^{n} b_k$$

$$\implies s_n + t_n = \sum_{k=1}^{n} (a_k + b_k), \quad c s_n = \sum_{k=1}^{n} c a_k$$

$$\implies s + t = \sum_{k=1}^{\infty} a_k + b_k, \quad c s = \sum_{k=1}^{\infty} c a_k$$

since $\lim_{n\to\infty}(s_n + t_n) = s + t, \quad \lim_{n\to\infty} c s_n = cs.$

$\square$

If the terms a_k in the series are all of the same sign, then $\{s_n\}$ is a monotone sequence and therefore convergent if and only if it is bounded. We will now consider series of *positive* terms

THEOREM 11.2.7. (THE COMPARISON TEST). *Suppose*

$$0 \le a_k \le b_k, \quad k = 1, 2, \ldots .$$

Then

(a) $\displaystyle\sum_{k=1}^{\infty} a_k$ *divergent* $\implies \displaystyle\sum_{k=1}^{\infty} b_k$ *divergent,*

(b) $\displaystyle\sum_{k=1}^{\infty} b_k$ *convergent* $\implies \displaystyle\sum_{k=1}^{\infty} a_k$ *convergent.*

PROOF: The statements (a) and (b) are equivalent. If $s_n = \displaystyle\sum_{k=1}^{n} a_k, \ t_n = \displaystyle\sum_{k=1}^{n} b_k,$ then $\{s_n\}$ unbounded $\implies \{t_n\}$ unbounded or equivalently $\{t_n\}$ bounded $\implies \{s_n\}$ bounded and, since both sequences are monotone, the theorem follows.

$\square$

Clearly the conclusion of the Comparison Test is valid if $0 \le a_k \le b_k$ holds only for k large.

EXAMPLE 11.2.8. $\sum_{k=2}^{\infty} \frac{1}{(\log k)^k}$ is convergent since

$$0 < \frac{1}{\log k} < \frac{1}{2}, \quad \text{if} \quad \log k > 2 \quad \text{i.e.} \quad k > e^2.$$

Thus

$$0 < \frac{1}{(\log k)^k} < \frac{1}{2^k}$$

implies

$$\sum_{k=2}^{\infty} \frac{1}{(\log k)^k} \quad \text{is convergent since}$$

$\sum_{k=0}^{\infty} \frac{1}{2^k}$ is convergent (Example 11.2.1, $x = \frac{1}{2}$).

$\square$

COROLLARY 11.2.9. (LIMIT FORM OF THE COMPARISON TEST). *Suppose* $a_k \geq 0, \ b_k > 0, \ k = 1, 2, \ldots$ *and*

$$\lim_{k \to \infty} \frac{a_k}{b_k} = L, \quad 0 \leq L < \infty.$$

Then

(a) $\sum_{k=1}^{\infty} b_k$ is convergent $\iff \sum_{k=1}^{\infty} a_k$ is convergent, if $L > 0$

(b) $\sum_{k=1}^{\infty} b_k$ is convergent $\implies \sum_{k=1}^{\infty} a_k$ is convergent, if $L = 0$.

PROOF: In case (a), $0 < \frac{1}{2} b_k < a_k < 2L b_k$ for k sufficiently large. Part (b) is left as an exercise.

$\square$

EXAMPLE 11.2.10. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent since (Example 11.2.3)

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$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ is convergent and

$$\lim_{k\to\infty} \frac{1/k^2}{1/k(k+1)} = \lim_{k\to\infty} \frac{k(k+1)}{k^2} = 1.$$

$\square$

COROLLARY 11.2.10 (RATIO FORM OF THE COMPARISON TEST). *Suppose that* $a_k > 0$, $b_k > 0$ *and*

$$\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k}$$

hold for all large k *and* $\sum_{k=1}^{\infty} b_k$ *convergent* $\implies \sum_{k=1}^{\infty} a_k$ *convergent.*

PROOF: If $\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k}$ holds for $k \geq N$, then $\frac{a_{k+1}}{b_{k+1}} \leq \frac{a_k}{b_k}$ for $k \geq N$ so that $\{\frac{a_k}{b_k}\}$ is a decreasing sequence, for $k \geq N$, and

$$\frac{a_k}{b_k} \leq M \quad \text{where} \quad M = \frac{a_N}{b_N}$$

which implies

$$0 < a_k \leq M b_k$$

and

$$\sum_{k=1}^{\infty} b_k \quad \text{convergent} \implies \sum_{k=1}^{\infty} M b_k \quad \text{convergent} \implies \sum_{k=1}^{\infty} a_k \quad \text{convergent},$$

by the Comparison Test.

$\square$

The simplest choice of comparison series is $b_k = c^k$, $c \geq 0$.
The geometric series $\sum\limits_{k=0}^{\infty} b_k = \sum\limits_{k=0}^{\infty} c^k$ is convergent if $0 \leq c < 1$
and divergent if $c \geq 1$. Here $\frac{b_{k+1}}{b_k} = c$ so that

$$\frac{a_{k+1}}{a_k} \leq c < 1 \Longrightarrow \sum\limits_{k=1}^{\infty} a_k \quad \text{convergent}$$

$$\frac{a_{k+1}}{a_k} \geq 1 \Longrightarrow \sum\limits_{k=1}^{\infty} a_k \quad \text{divergent.}$$

In the second statement, the roles of a_k, b_k have been interchanged.
If the condition $\frac{a_{k+1}}{a_k} \leq c < 1$ is replaced by $\frac{a_{k+1}}{a_k} < 1$ it cannot
be concluded that $\sum\limits_{k=1}^{\infty} a_k$ is convergent. Why?

It is convenient to formulate this comparison with the geometric
series in limit form.

COROLLARY 11.2.11 (LIMIT FORM OF THE RATIO COMPARISON
TEST). *Suppose* $a_k > 0$ *and* $\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = c$, *then*

(a) $0 \leq c < 1 \Longrightarrow \sum\limits_{k=1}^{\infty} a_k$ *is convergent*

(b) $1 < c \Longrightarrow \sum\limits_{k=1}^{\infty} a_k$ *is divergent*

(c) $1 \Longrightarrow ?$, *the test fails.*

PROOF: This Corollary is a method of comparison with the geometric
series $\sum\limits_{k=0}^{\infty} x^k$ which is convergent if $|x| < 1$ and divergent for
$|x| \geq 1$. In case (a) $\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = c < 1$. Thus, if we choose
$x, c < x < 1$ and $b_k = x^k$,

$$\frac{a_{k+1}}{k} < x = \frac{b_{k+1}}{b_k}$$

so that $\sum\limits_{k=1}^{\infty} x^k = \sum\limits_{k=1}^{\infty} b_k$ convergent $\Longrightarrow \sum\limits_{k=1}^{\infty} a_k$ convergent.

In case (b) $\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=c>1$ and if $c>x>1$ then $\frac{a_{k+1}}{a_k}>x=\frac{b_{k+1}}{b_k}$ and since $\sum_{k=1}^{\infty}x^k=\sum_{k=1}^{\infty}b_k$ is divergent we also have $\sum_{k=1}^{\infty}a_k$ is convergent. Note that in this case we have interchanged the roles of a_k,b_k in Corollary 11.2.10.

To see that the test fails if $c=1$, consider the series $\sum_{k=1}^{\infty}k,\ \sum_{k=1}^{\infty}\frac{1}{k^2}$. The first series is divergent and the second is convergent but both satisfy $\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=1$.

$\square$

EXAMPLE 11.2.12. The series $\sum_{k=1}^{\infty}\frac{2^k}{k^2}$ is diverget since $\frac{2^{k+1}}{(k+1)^2}/\frac{2^k}{k^2}=$ $2(1+\frac{1}{k})^{-2}\xrightarrow[k\to\infty]{}2>1$. Similarly $\sum_{k=1}^{\infty}\frac{k^3}{2^k}$ is convergent; in this case the ratio of two successive terms $\frac{(k+1)^3}{2^{k+1}}/\frac{k^3}{2^k}=\frac{1}{2}\left(1+\frac{1}{k}\right)^3\xrightarrow[k\to\infty]{}\frac{1}{2}<1$.

EXAMPLE 11.2.13. If $x\geq 0$, $\sum_{k=0}^{\infty}\frac{x^k}{k!}$ is convergent, since $\frac{x^{k+1}}{(k+1)!}/\frac{x^k}{k!}=$ $\frac{x}{k+1}\xrightarrow[k\to\infty]{}0<1$. It follows that $\sum_{k=1}^{\infty}\frac{x^k}{k!}$ is convergent for all $x\in\mathbb{R}$. Since

$$|\sum_{k=m}^{m}\frac{x^k}{k!}|\leq\sum_{k=m}^{m}\frac{|x|^k}{k!},\quad\text{by the triangle inequality,}$$

and, since $\sum_{k=0}^{\infty}\frac{|x|^k}{k!}$ is convergent, the Cauchy Criterion implies $\sum_{k=0}^{\infty}\frac{x^k}{k!}$ is convergent.

To increase our stock of comparison series we prove *The Integral Test* for series of positive decreasing terms.

LEMMA 11.2.14.. *Suppose* f *is decreasing on* $[1,\infty)$. *Then*

$$f(2)+f(3)+\cdots+f(n)\leq\int_1^n f(x)dx\leq f(1)+f(2)+\cdots+f(n-1).$$

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PROOF: Let $P = \{1, 2, \ldots, n\}$.

Then

$L(P, f) = f(2) + f(3) + \cdots + f(n)$

$U(P, f) = f(1) + f(2) + \cdots + f(n - 1)$

and therefore the Lemma

follows from the definition of $\int_1^n f$.

□
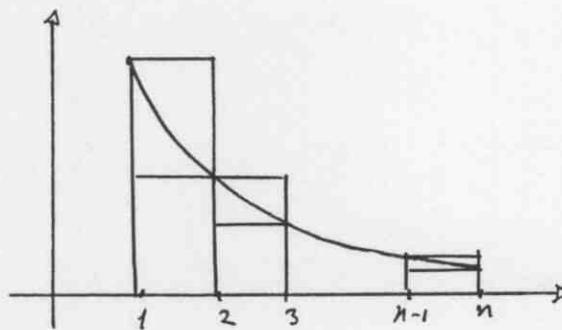
PROPOSITION 11.2.15. (THE INTEGRAL TEST). *Suppose* $f \geq 0$ *is decreasing on* $[1, \infty)$. *Then* $\sum_{k=1}^{\infty} f(k)$ *is convergent* $\iff$ $\lim_{n \to \infty} \int_1^n f(x)dx$ *exists.*

PROOF: If $s_n = \sum_{k=1}^{n} f(k)$, then from the Lemma

$$s_n - f(1) \leq \int_1^n f \leq s_{n-1}.$$

Since both $\{s_n\}$ and $\{\int_1^n f\}$ are increasing sequences, they converge if and only if they are bounded and the inequality shows they are either both bounded or neither of them is.

PROPOSITION 11.2.15. *The series* $\sum_{k=1}^{\infty} \frac{1}{k^p}$ *is*

(a) *Convergent if* $p > 1$,

(b) *Divergent if* $p \leq 1$.

PROOF: If $p \leq 0$, then $\frac{1}{k^p} \geq 1$, $k = 1, 2, \ldots$ so that $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is divergent by the Comparison Test, since $\sum_{k=1}^{\infty} 1$ is divergent.

If $p > 0$, then $f(x) = \frac{1}{x^p}$ is decreasing on $(0, \infty)$ and

$$\int_1^n \frac{1}{x^p} \, dx = \begin{cases} \frac{1}{1-p} x^{1-p}|_1^n, & p \neq 1 \\ \log x|_1^n, & p = 1 \end{cases}$$

$$= \begin{cases} \frac{1}{1-p} (n^{1-p} - 1), & p \neq 1 \\ \log n, & p = 1. \end{cases}$$

Therefore $\lim_{n \to \infty} \int_1^n \frac{1}{x^p} \, dx = \frac{1}{p-1}$, if $p > 1$ and the limit does not exist if $p \leq 1$ and the Proposition follows from the Integral Test.

NOTATION. Before proceeding, we introduce the *Landau* O, o *notation*. Let $f(x)$ and $g(x)$ be defined for x near a (we do not exclude $a = \infty$). We say

$$f(x) = O\big(g(x)\big), \quad \text{as} \quad x \to a$$

if $\frac{f(x)}{g(x)}$ is bounded for x sufficiently close to a;

$$f(x) = o\big(g(x)\big), \quad \text{as} \quad x \to a$$

if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

Thus, in the case of sequences,

$$a_k = O(b_k), \quad k \to \infty$$

means $\left| \frac{a_k}{b_k} \right| \leq M$ and

$$a_k = o(b_k), \quad k \to \infty$$

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means $\lim_{n \to \infty} \frac{a_k}{b_k} = 0$.

LEMMA 11.2.17. *For any real number* α,

$$(1+x)^\alpha = 1 + \alpha x + O(x^2), \quad \text{as} \quad x \to 0.$$

PROOF: From Taylor's Theorem,

$$(1+x)^\alpha = 1 + \alpha x + r_1(x)$$

where $r_1(x) = \frac{\alpha(\alpha-1)}{2!}(1+c)^{\alpha-2}x^2$, for some c between 0 and x. Therefore $r_1(x) = O(x^2)$, as $x \to 0$.

PROPOSITION 11.2.18 (RAABE'S TEST). *(a) Suppose* $a_k > 0$ *and*

$$\frac{a_{k+1}}{a_k} = 1 - \frac{p}{k} + o(\frac{1}{k}), \quad \text{as} \quad k \to \infty.$$

Then $\sum_{k=1}^{\infty} a_k$ *is convergent if* $p > 1$, *divergent if* $p < 1$.
  (b) Suppose $a_k > 0$ *and*

$$\frac{a_{k+1}}{a_k} = 1 - \frac{p}{k} + O(\frac{1}{k^2}), \quad \text{as} \quad k \to \infty.$$

Then $\sum_{k=1}^{\infty} a_k$ *is convergent if* $p > 1$, *divergent if* $p \le 1$.

We will see that this test, roughly speaking, deals with the case $c = 1$ where the Ratio Test fails and that it is the Comparison Test in the ratio form using the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ as comparison series. Observe that Part (a) of the Proposition states that if $a_k > 0$ and the difference between $\frac{a_{k+1}}{a_k}$ and $1 - \frac{p}{k}$ tends to zero a little faster than the sequence $\{\frac{1}{k}\}$, $k \to \infty$, then $\sum_{k=1}^{\infty} a_k$ is convergent

if $p > 1$, divergent if $p < 1$ and the case $p = 1$ is left in doubt. Part (b) says that if the difference between $\frac{a_{k+1}}{a_k}$ and $1 - \frac{p}{k}$ tends to zero at least as fast as the sequence $\{\frac{1}{k^2}\}$, $n \to \infty$, then we have the same conclusion and further, even if $p = 1$, we can say $\sum_{k=1}^{\infty} a_k$ is divergent.

PROOF: To prove Part (a), recall that $\sum_{k=1}^{\infty} \frac{1}{k^q}$ is convergent if $q > 1$, divergent if $q \leq 1$. Let $b_k = \frac{1}{k^q}$, so that

$$\frac{b_{k+1}}{b_k} = \frac{k^q}{(k+1)^q} = (1 + \frac{1}{k})^{-q} = 1 - \frac{q}{k} + O(\frac{1}{k^2}),$$

by Lemma 11.2.16. Now,

$$\frac{a_{k+1}}{a_k} - \frac{b_{k+1}}{b_k} = \frac{q-p}{k} + o(\frac{1}{k}) + O(\frac{1}{k^2})$$
$$= \frac{q-p}{k} + o(\frac{1}{k}).$$

If $p > 1$, choose $q \in (1, p)$ so that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^q}$ is convergent and $\frac{a_{k+1}}{a_k} < \frac{b_{k+1}}{b_k}$, for all large k, and hence $\sum_{k=1}^{\infty} a_k$ is convergent.

If $p < 1$, choose $q = 1$ so that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ is divergent and $\frac{a_{k+1}}{a_k} > \frac{b_{k+1}}{b_k}$, for all large k and therefore $\sum_{k=1}^{\infty} a_k$ is divergent.

To prove Part (b), first observe that we need only consider $p = 1$ since the other parts of this assertion are implied by Part (a). Now

$$\frac{a_{k+1}}{a_k} = 1 - \frac{1}{k} + O(\frac{1}{k^2}), \quad k \to \infty$$

implies that, for some real number A,

$$\frac{a_{k+1}}{a_k} \geq 1 - \frac{1}{k} + \frac{A}{k^2} .$$

Now consider $b_k = \frac{1}{k+A-1}$ so that

(B) $\quad \frac{b_{k+1}}{b_k} = \frac{k+A-1}{k+A} = 1 - \frac{1}{k+A} = 1 - \frac{1}{k}(1 + \frac{A}{k})^{-1}.$

Now, since

$$1 - \frac{A^2}{k^2} \leq 1 \qquad \text{for all} \quad k \quad \text{implies}$$

$$1 - \frac{A}{k} \leq (1 + \frac{A}{k})^{-1}, \quad \text{for all large} \quad k, \quad \text{so that}$$

(C) $\quad 1 - \frac{1}{k}(1 + \frac{A}{k})^{-1} \leq 1 - \frac{1}{k}(1 - \frac{A}{k}) = 1 - \frac{1}{k} + \frac{A}{k^2}$

and therefore (A), (B), (C) $\implies \frac{a_{k+1}}{a_k} \geq \frac{b_{k+1}}{b_k}$, for all large k and

$\sum_{k=1}^{\infty} b_k$ divergent $\implies \sum_{k=1}^{\infty} a_k$ divergent.

EXAMPLE 11.2.19. The series $\sum_{k=1}^{\infty} \frac{1.3.5...(2k-1)}{2.4.6...2k}$ is divergent by

Raabe's Test, since

$$\frac{a_{k+1}}{a_k} = \frac{2k+1}{2k+2} = \frac{1 + \frac{1}{2k}}{1 + \frac{1}{k}} = (1 + \frac{1}{2k})(1 + \frac{1}{k})^{-1}$$

$$= (1 + \frac{1}{2k})(1 - \frac{1}{k} + O(\frac{1}{k^2})), \quad k \to \infty$$

$$= 1 - \frac{1/2}{k} + O(\frac{1}{k^2}), \quad k \to \infty,$$

and $\frac{1}{2} < 1.$

EXAMPLE 11.2.20. Consider the series

$$\sum_{k=1}^{\infty} \frac{(a+1)(a+2)...(a+k)}{(b+1)(b+2)...(b+k)} .$$

The Ratio Test Fails here (Critical case $c = 1$). However

$$\frac{a_{k+1}}{a_k} = \frac{(a+k+1)}{(b+k+1)} = \frac{(1+\frac{a+1}{k})}{(1+\frac{b+1}{k})} = (1+\frac{a+1}{k})(1+\frac{b+1}{k})^{-1}$$

$$= (1+\frac{a+1}{k})(1-\frac{b+1}{k}+O(\frac{1}{k^2}))$$

$$= 1 - \frac{b-a}{k} + O(\frac{1}{k^2})$$

so that the series is convergent if $b-a > 1$ and divergent if $b-a \leq 1$, by Raabe's Test.

EXAMPLES 11.2.21. In the following examples, supply the missing details.

(1) $\sum_{k=1}^{\infty} \frac{k}{k+2}$ is divergent.

(2) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ is convergent.

(3) $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ is divergent.

(4) $\sum_{k=1}^{\infty} \frac{1}{1+\log k}$ is divergent.

(5) $\sum_{k=2}^{\infty} \frac{1}{k(\log k)^p}$ is divergent if $p \leq 1$ and convergent if $p > 1$

(6) $\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$ is convergent.

(7) $\sum_{n=1}^{\infty} \frac{n^3}{n^4+3}$.

(8) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$.

(9) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent.

(10) $\sum_{k=1}^{\infty} \frac{k(2k+1)}{k!}$ is convergent.

(11) $\sum_{k=0}^{\infty} \frac{k^2}{3^k}$ is convergent.

(12) $\sum_{k=0}^{\infty} \frac{k!}{10^k}$ is divergent.

(13) $\sum_{k=1}^{\infty} \frac{2^k}{k(k+1)}$ is divergent.

(14) $\sum_{k=1}^{\infty} \left[\frac{1.3.5...(2k-1)}{2.4.6...2k}\right]^p$ is convergent if $p > 2$ and divergent if

$p \le 2$. [*Hint*: Show that $\frac{a_{k+1}}{a_k} = 1 - \frac{p}{2k} + O(\frac{1}{k^2})$, $k \to \infty$.

Compare with Example 11.2.19.]

(15) $\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}$.

(16) $\sum_{k=1}^{\infty} \sin(\frac{1}{k^2})$ is convergent.

§11.3 Absolute and Conditional Convergence.

We now consider series where the condition $a_k \ge 0$ is no longer assumed. The series $\sum_{k=1}^{\infty} a_k$ is said to be *absolutely convergent* if $\sum_{k=1}^{\infty} |a_k|$ is convergent.

THEOREM 11.3.1.. *An absolutely convergent series* $\sum_{k=1}^{\infty} a_k$ *is convergent and satisfies* $|\sum_{k=1}^{\infty} a_k| \le \sum_{k=1}^{\infty} |a_k|$.

PROOF: Since $|\sum_{k=n}^{m} a_k| \le \sum_{k=n}^{m} |a_k|$, by the Triangle Inequality, the Cauchy Criterion shows that

$$\sum_{k=1}^{\infty} |a_k| \quad \text{convergent} \quad \Rightarrow \sum_{k=1}^{\infty} a_k \quad \text{convergent}.$$

Moreover, taking $n = 1$ and the limit $m \to \infty$, we find $|\sum_{k=1}^{\infty} a_k| \le \sum_{k=1}^{\infty} |a_k|$.

□

There are series which are convergent but not absolutely convergent. For example,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \log 2$$

(see Problem 10.7) is a convergent series but not absolutely convergent, since $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent.

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The sum of an absolutely convergent series is not altered by a rearrangement of the terms in the series. In fact a series is absolutely convergent if and only if it has this property. For example

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log\, 2$$

but

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2}\, \log\, 2.$$

Here we are taking two positive terms followed by one negative and, even though every term of the original series is ultimately included, the sum is altered. To see this, let $\{s_n\}, \{t_n\}$ be the partial sums of the original series and the rearranged series respectively. Then

$$t_{3n} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n}.$$

Now

$$s_{4n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n}$$

$$\frac{1}{2}\, s_{2n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{12} - \cdots + \frac{1}{4n-2} - \frac{1}{4n}$$

so that

$$t_{3n} = s_{4n} + \frac{1}{2}s_{2n} \xrightarrow[n \to \infty]{} \log\, 2 + \frac{1}{2}\, \log\, 2 = \frac{3}{2}\, \log\, 2$$

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since $\lim_{n\to\infty} s_n = \log 2$. Also

$$t_{3n+1} = t_{3n} + \frac{1}{4n+1} \xrightarrow[n\to\infty]{} \frac{3}{2} \log 2$$

$$t_{3n+2} = t_{3n} + \frac{1}{4n+1} + \frac{1}{4n+3} \xrightarrow[n\to\infty]{} \frac{3}{2} \log 2$$

since $\lim_{n\to\infty} \frac{1}{4n+1} = \lim_{n\to\infty} \frac{1}{4n+3} = 0$ and therefore

$$\lim_{n\to\infty} t_n = \frac{3}{2} \log 2$$

as asserted.

In fact, if you reflect on it for a while you will see that a non-absolutely convergent series may be given *any* sum by rearrangement and may even be made to diverge to $\pm\infty$.

We may test for absolute convergence of $\sum_{k=1}^{\infty} a_k$ by applying the tests of the preceding section to $\sum_{k=1}^{\infty} |a_k|$.

EXAMPLE 11.3.2. The series $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ is absolutely convergent by the Comparison Test, since

$$0 \le \left| \frac{\sin k}{k^2} \right| \le \frac{1}{k^2}.$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent.

EXAMPLE 11.3.3. The series $\sum_{k=1}^{\infty} (-1)^k \frac{k^2}{2^k}$ is absolutely convergent, by the Ratio Test, since

$$\frac{|(-1)^{k+1}(k+1)^2/2^{k+1}|}{|(-1)^k k^2/2^k|} = \frac{1}{2}(1 + \frac{1}{k})^2 \xrightarrow[k\to\infty]{} \frac{1}{2} < 1.$$

There are several rather sophisticated tests for convergence of series which are not necessarily absolutely convergent. Here we confine

our attention to *alternating series* of the form $\sum_{k=1}^{\infty}(-1)^{k+1}a_k,$
$a_k > 0,$ for which a simple but effective test is available.

THEOREM 11.3.4.. *(Leibniz' Altering Series Test).* Suppose $\{a_n\}$
is decreasing. Then

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k \quad \text{is convergent} \iff \lim_{n\to\infty} a_n = 0.$$

CAUTION: The monotonicity assumption is essential. For example,
the series $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2}\ldots$ is divergent. Why?

PROOF: The condition $\lim_{n\to\infty} a_n = 0$ is clearly necessary for the
convergence of the series. To see that in the present circumstances it
is also sufficient, consider $s_n = \sum_{k=1}^{\infty}(-1)^{k+1}a_k.$ Then

$$s_{2n} = a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n}.$$

Therefore

$$s_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}),$$

a sum of positive terms so that $\{S_{2n}\}$ is a positive increasing se-
quence. Moreover

$$s_{2n} = a_1 - (a_2 - a_3) + \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}$$

$$\leq a_1$$

so that $\{s_{2n}\}$ is a bounded sequence and

$$s = \lim_{n\to\infty} s_n \quad \text{exists.}$$

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Since $s_{2n+1} = s_{2n} + \frac{1}{2n+1}$, $s = \lim_{n \to \infty} s_{2n+1}$ also and therefore

$$s = \lim_{n \to \infty} s_n \quad \text{exists.}$$

REMARKS.

(i) The proof shows that, under the conditions of Leibniz' Test,

$$0 \le \sum_{k=1}^{\infty} (-1)^{k+1} a_k \le a_1.$$

(ii) Under the same conditions, the error $\sum_{k=N+1}^{\infty} (-1)^{k+1} a_k$ is truncating the series at N terms is of the same sign as and smaller in magnitude than the first term neglected.

EXAMPLE 11.3.5. The alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$,

$\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{\log k}$ are all convergent, by the Leibniz Test. □

We may adapt Raabe's Test so that it is useful in testing some alternating series when used in conjunction with Leibniz' Test.

LEMMA 11.3.6. (RAABE'S TEST FOR SEQUENCES). *Suppose* $a_k > 0$ *and*

$$\frac{a_{k+1}}{a_k} = 1 - \frac{p}{k} + o\left(\frac{1}{k}\right), \quad k \to \infty.$$

Then

(a) $p > 0 \Longrightarrow \{a_n\}$ *is ultimately monotone and* $\lim_{n \to \infty} a_n = 0$.

(b) $p < 0 \Longrightarrow \{a_k\}$ *is ultimately monotone and* $\lim_{n \to \infty} a_k = \infty$.

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The proof of this lemma is left as an exercise (Problem 11.27).

$\square$

PROPOSITION 11.3.7. (RAABE'S TEST FOR ALTERNATING SERIES). *Suppose* $a_k > 0$ *and*

$$\frac{a_{k+1}}{a_k} = 1 - \frac{p}{k} + o(\frac{1}{k}), \quad k \to \infty.$$

Then

$$p > 0 \Longrightarrow \sum_{k=1}^{\infty} (-1)^{k+1} a_k \quad \text{is convergent}$$

$$p > 0 \Longrightarrow \sum_{k=1}^{\infty} (-1)^{k+1} a_k \quad \text{is divergent}$$

and if $p = 0$, *the series may be either convergent or divergent.*

$\square$

The assertions in the cases $p > 0$, $p < 0$ follow from the Lemma and Leibniz' Test. The doubtful case $p = 0$ is left to problems.

EXAMPLE 11.3.8. The series

$$\sum_{k=1}^{\infty} (-1)^k \frac{(a+1)(a+2)\ldots(a+k)}{(b+1)(b+2)\ldots(b+k)}$$

is absolutely convergent if $b - a > 1$ (see Example 11.2.20) conditionally convergent if $1 \geq b - a > 0$ and divergent if $b - a \leq 0$. The assertion in the case $b = a$ is obvious (Why?). The cases $1 \geq b - a > 0$, $b - a < 0$ follow from Proposition 11.3.7; the details of the calculation are the same as in Example 11.2.20. $\square$

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§11.4. Power Series.

A power series is an expresion of the form

$$\sum_{k=0}^{\infty} a_k(x-a)^k.$$

For example, if a function $f(x)$ has derivatives of all orders at $x=a$ and $p_n(x)$ is the corresponding Taylor polynomial about $x=a$, it is natural to enquire if $\lim_{n\to\infty} p_n(x) = f(x)$ for some values of x. This is equivalent to $\lim_{n\to\infty} r_n(x) = 0$ or

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

This power series is called the Taylor series of $f(x)$ about $x=a$. Clearly it converges to $f(a)$ if $x=a$ and converges to $f(x)$ for all x, if f is a polynomial. However, even if the Taylor series converges for $x \neq a$, it may not converge to the function $f(x)$. For example if $f(x) = e^{-\frac{1}{x^2}}$, $x \neq 0$, $f(0) = 0$, then $f^{(k)}(0) = 0$, $k = 0,1,2,\ldots$. The Taylor series has sum 0 for all x but $f(x) \neq 0$ if $x \neq 0$.

THEOREM 11.4.1.. *For each power series* $\sum_{k=0}^{\infty} a_k(x-a)^k$, *there exists an* R, $0 \leq R \leq \infty$, *such that the series is absolutely convergent if* $|x-a| < R$ *and divergent if* $|x-a| > R$.

PROOF $(a=0)$: It suffices to prove this when $a=0$ since the general case reduces to this by the substitution $X = x-a$. Suppose $\sum_{k=0}^{\infty} a_k x_0^k$ is convergent for some $x_0 \neq 0$. Then $|a_k x_0^k| \leq M$ for

some M (in fact $\lim_{k \to \infty} a_k x_0^k = 0$). Since

$$|a_k x^k| = |a_k x_0^k| \, |\frac{x}{x_0}|^k \leq M|\frac{x}{x_0}|^k.$$

If $|x| < |x_0|$, the geometric series $\sum_{k=0}^{\infty} M|\frac{x}{x_0}|^k$ is convergent. The Comparison Test therefore implies that $\sum_{k=0}^{\infty} a_k x^k$ is absolutely convergent if $|x| < |x_0|$. This also shows that if $\sum_{k=0}^{\infty} a_k x_1^k$ is divergent, then $\sum_{k=0}^{\infty} a_k x^k$ is divergent if $|x| > |x_1|$. The existence of R as asserted follows.

$\square$

The number R is called the *radius of convergence* of the power series.

EXAMPLE 11.4.2.

(i) $\sum_{k=0}^{\infty} k! x^k$ is convergent if $x = 0$, divergent if $|x| > 0 \, (R = 0)$ by the Ratio Test since

$$\frac{|(k+1)! x^{k+1}|}{|k! x^k|} = (k+1)|x| \to \begin{cases} \infty, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

(ii) $\sum_{k=0}^{\infty} k x^k$ is absolutely convergent if $|x| < 1$ and divergent if $|x| \geq 1$. Here

$$\frac{|(k+1) x^{k+1}|}{|k x^k|} = (1 + \frac{1}{k})|x| \xrightarrow[k \to \infty]{} |x|,$$

so the Ratio Test shows that the series is absolutely convergent if $|x| < 1$ and is not absolutely convergent if $|x| > 1$. At $x = \pm 1$ the series is divergent since $\{k x^k\}$ is unbounded.

(iii) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ is abolutely convergent for all x by the Ratio Test

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since

$$\frac{|x^{k+1}/(k+1)!|}{|x^k/k!|} = \frac{|x|}{k+1} \xrightarrow[k \to \infty]{} 0 < 1.$$

The power series may be divergent at one or both of $a \pm R$ or it may be conditionally convergent at one or both or it may be absolutely convergent at both points. For example, the series (ii) is divergent at $x = \pm 1$.

(iv) $\sum_{k=1}^{\infty} \frac{x^k}{k}$ has radius of convergence $R = 1$ (Ratio Test) it is conditionally convergent at $x = -1$ (Leibniz Text) and divergent at $x = 1$.

(v) $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$ has radius of convergence $R = 1$ (Ratio Test) and it is absolutely convergent at $x = \pm 1$ (Integral Test).

THEOREM 11.4.3. *The power series*

$$(i) \quad \sum_{k=0}^{\infty} a_k x^k, \qquad (ii) \quad \sum_{k=1}^{\infty} k a_k x^{k-1}$$

have the same radius of convergence.

The series (ii) is obtained by formally differentiating (i) term-by-term. It follows that the series

(iii) $\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$ also has the same radius of convergence as (i), which may be obtained by formally differentiating (iii).

PROOF OF THEOREM 11.4.3: Since

$$|a_k x^k| = |x|\, |a_k x^{k-1}| \leq |x|\, |k a_k x^{k-1}|,$$

it follows from the Comparison Test that if (ii) is Absolutely Convergent, then so also is (i). Conversely suppose $\sum_{k=0}^{\infty} a_k x_0^k$ is conver-

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gent, $x_0 \neq 0$. Then $|a_k x_0^k| \leq M$ and, if $|x| < |x_0|$,

$$|k a_k x^{k-1}| \leq |a_k x_0^k| \frac{a}{|x_0|} k \left|\frac{x}{x_0}\right|^{k-1}$$

$$\leq \frac{M}{|x_0|} k \left|\frac{x}{x_0}\right|^{k-1}.$$

The Ratio Test shows $\sum_{k=1}^{\infty} k \left|\frac{x}{x_0}\right|^{k-1}$ is convergent, if $\left|\frac{x}{x_0}\right| < 1$ (see Example 11.4.2 (ii)), so the Comparison Test shows $\sum_{k=1}^{\infty} k a_k x^{k-1}$ is absolutely convergent if $|x| < |x_0|$. Thus (i) and (ii) have the same radius of convergence.

$\square$

PROPOSITION 11.4.4. If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R > 0$, then f is continuous on $(-R, R)$.

PROOF: Let $x_0 \in (-R, R)$. We must prove $\lim_{x \to x_0} f(x) = f(x_0)$.

Choose r so that

$|x_0| < r < R$.
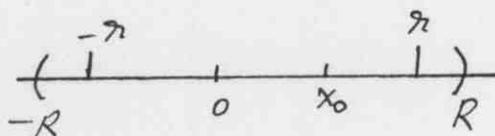
Then $\sum_{k=0}^{\infty} |a_r| r^k$ is convergent
and, if $\varepsilon > 0$, we may choose N so that

$$\sum_{k=N+1}^{\infty} |a_r| r^k < \frac{\varepsilon}{3}.$$

Then

$$f(x) - f(x_0) = \sum_{k=0}^{\infty} a_k x^k - \sum_{k=0}^{\infty} a_k x_0^k$$

$$= \sum_{k=0}^{N} a_k x^k - \sum_{k=0}^{N} a_k x_0^k + \sum_{k=N+1}^{\infty} a_k x^k - \sum_{k=N+1}^{\infty} a_k x_0^k$$

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so that

$$|f(x) - f(x_0)| \leq |\sum_{k=0}^{N} a_k x^k - \sum_{k=0}^{N} a_k x_0^k| + \sum_{k=N+1}^{\infty} |a_k x^k| + \sum_{k=N+1}^{\infty} |a_k x_0|^k.$$

Now the polynomial $\sum_{k=0}^{N} a_k x^k$ is continuous, so we may choose $\delta > 0$ so that $\delta < \min\{|x_0 + r|, |x_0 - r|\}$ and $|x - x_0| < \delta$ implies

$$|\sum_{k=0}^{N} a_k x^k - \sum_{k=0}^{N} a_k x_0^k| < \frac{\varepsilon}{3} \quad \text{and}$$

$$\sum_{k=N+1}^{\infty} |a_k x^k| \leq \sum_{k=N+1}^{\infty} |a_k| r^k < \frac{\varepsilon}{3},$$

so that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

$\square$

THEOREM 11.4.5. Suppose $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R > 0$. Then

(a) $\int_0^x f = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}, \quad |x| < R$

(b) $f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}, \quad |x| < R.$

PROOF: We have see (Theorem 11.4.3) that all three series have the same radius of convergence. Since f is continuous on $(-R, R)$ (Proposition 11.4.4), $\int_0^x f$ exists. Furthermore

$$|\int_0^x f - \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1}| = |\int_0^x (\sum_{k=n+1}^{\infty} a_k t^k) dt| \leq |\int_0^x |\sum_{k=n+1}^{\infty} a_k t^k| dt|$$

since the integral on the right also exists by Proposition 11.4.4. If $|x| \leq r < R$, then $|t| \leq |x|$ implies

$$|\sum_{k=n+1}^{\infty} a_k t^k| \leq \sum_{k=n+1}^{\infty} |a_k t^k| \quad \text{(the series is absolutely convergent)}$$

$$\leq \sum_{k=n+1}^{\infty} |a_k| r^k$$

$$\xrightarrow[n\to\infty]{} 0 \quad \text{since} \quad \sum_{k=0}^{\infty} |a_k| r^k \quad \text{is convergent.}$$

Therefore $\quad |\int_0^x |\sum_{k=n+1}^{\infty} a_k t^k| dt \leq |x| \sum_{k=n+1}^{\infty} |a_k| r^k \xrightarrow[k\to\infty]{} 0 \quad$ and

$$|\int_0^x f - \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1}| \xrightarrow[k\to\infty]{} 0$$

so that $\quad \int_0^x f = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$. This proves (a).

To prove (b) we see from (a), if $\quad g(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$, $|x| < R$, then $\quad f(x) = a_0 + \int_0^x g \quad$ and therefore $\quad f'(x) = g(x)$, as asserted.

$\square$

COROLLARY 11.4.6. *Suppose* $\quad f(x) = \sum_{k=0}^{\infty} a_k x^k \quad$ *has radius of convergence* $R > 0$. *Then*

$$a_k = \frac{f^{(k)}(0)}{k!}, \quad k = 0, 1, \ldots$$

so that the power series is the Taylor series of f.

This can be seen by observing that Theorem 11.4.5 (b) implies that f has derivatives of all orders given by $f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$,

$f''(x) = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}$, and so forth.

A convergent power series is therefore the Taylor series of its sum f.

The following examples might all be deduced from Taylor's Theorem by showing that the remainder $r_n(x)$ satisfies $\lim_{n \to \infty} r_n(x) = 0$ in each case. It is however easier to obtain them by substitutions, differention and integration of the geometric series.

EXAMPLE 11.4.7.

(i) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots,\quad |x| < 1,$

$$\text{since} \quad 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}, \quad x \neq 1.$$

(ii) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots,\quad |x| < 1.$

Replace x by $-x$ in (i).

(iii) $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots,\quad |x| < 1.$

Integrate in (ii) from 0 to x.

(iv) $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots,\quad |x| < 1.$

Differentiate (i).

(v) $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots,\quad |x| < 1.$

Replace x by x^2 in (ii) and integrate from 0 to x.

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(vi) $\tanh^{-1}x = \frac{1}{2}\log\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots, \quad |x| < 1.$

Replace x by x^2 in (i) and integrate from

0 to x (see Problem 7.38(c).

Alternatively, replace x by $-x$ in (iii),

subtract from (iii) and divide by 2.

Taylor's Theorem for $\log(1+x)$ shows that $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = $

log 2 (Problem 10.7). This also follows from Example 11.4.7 (iii)

even though this formula was only established for $|x| < 1$. Recall

that in an alternating series of decreasing terms the truncation error

is less than the first term neglected. Thus, from (iii)

$$|\log(1+x) - \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}x^k| \le \frac{x^{n+1}}{n+1},$$

$$< \frac{1}{n+1}, \quad \text{if} \quad 0 \le x < 1.$$

Take the limit, $x \to 1-$ on the left to obtain

$$|\log 2 - \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}| \le \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0$$

so that

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\dots.$$

A similar argument may be used on (v) to deduce

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots.$$

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EXAMPLE 11.4.8. The well-known identity

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad x \in \mathbb{R},$$

may be established easily from Taylor's Theorem (Problem 11.23). An alternative method of proving this is to observe, from Theorem 11.4.5, that if

$$E(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots,$$

then $E'(x) = E(x)$, $E(0) = 1$. Thus

$$0 = e^{-x}E'(x) - e^{-x}E(x) = \frac{d}{dx}\left[e^{-x}E(x)\right]$$

and therefore $e^{-x}E(x)$ is constant and in fact equal to 1 (set $x = 0$). Therefore we have $E(x) = e^x$.

EXAMPLE 11.4.9. Taylor's Theorem may also be used (Problem 11.23) to prove

$$\sin x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)!}, \quad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

An alternative proof of this is as follows. Let

$$S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Both series converge for all $x \in \mathbb{R}$ and, from Theorem 11.4.5, may be formally differentiated term-by-term to obtain

$$S'(x) = C(x), \quad C'(x) = -S(x)$$

and $S(0) = 0$, $C(0) = 1$.

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From here we find, on differentiation, that

$$g(x) = \big(S(x) - \sin x\big)^2 + \big(C(x) - \cos x\big)^2$$

satisfies $g'(x) = 0$, $g(0) = 0$, so that $g(x) = 0$ for all x. Hence

$$S(x) = \sin x, \quad C(x) = \cos x.$$

§11.5. Euler's Formula and de Moivre's Theorem.

The preceding discussion may be used to motivate an elegant unification of the transcendental functions $e^x, \cos x, \sin x$ known as Euler's Formula. But first we must consider the complex numbers $\mathbb{C}$. The complex number i satisfies $i^2 = -1$ and all complex numbers are of the form $a + ib$ where a and b are real. We consider $\mathbb{R} \subset \mathbb{C}$ in the sense that we may identify $a \in \mathbb{R}$ with the complex number $a + i0$ and may unambiguously write $a = a + i0$.

The algebraic operations $+$ and $\cdot$ on $\mathbb{C}$ are defined by

$$(a + ib) + (c + id) = (a + c) + i(b + d),$$

$$(a + ib) \cdot (c + id) = (ac - bd) + i(ad + bc).$$

In particular

$$(a + ib) \cdot (a - ib) = a^2 + b^2,$$

and we may take

$$\frac{1}{a + ib} = \frac{a - ib}{(a + ib)(a - ib)} = \frac{1}{a^2 + b^2}(a - ib).$$

We define $e^{i\theta}, \theta \in \mathbb{R}$, by *Euler's Formula*

$$e^{i\theta} = \cos\,\theta + i\,\sin\,\theta.$$

The motivation for this definition is that if we replace x by $i\theta$ in the formula

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots, \quad x \in \mathbb{R}$$

we obtain

$$e^{i\theta} = i + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$

$$= \cos\,\theta + i\,\sin\,\theta.$$

This is not a proof of Euler's formula, but simply a formal manipulation which motivates the formula as a *definition* of $e^{i\theta}$, $\theta \in \mathbb{R}$.

To further justify the definition we should check that $e^{i\theta}$ behaves as an exponential should. First $e^{i0} = \cos\,0 + i\,\sin\,0 = 1 + i0 = 1 = e^0$. Also

$$e^{i(x+y)} = \cos(x+y) + i\,\sin(x+y)$$

$$= (\cos\,x\,\cos\,y - \sin\,x\,\sin\,y) + i(\sin\,x\,\cos\,y + \cos\,x\,\sin\,y)$$

$$= (\cos\,x + i\,\sin\,x) \cdot (\cos\,y + i\,\sin\,y)$$

$$= e^{ix}e^{iy}, \quad x, y \in \mathbb{R}.$$

From this we readily deduce *de Moivre's Theorem*

$$(\cos\ x + i\ \sin\ x)^n = \cos\ nx + i\ \sin\ nx$$

for $n = 0, 1, 2, \ldots$. The theorem is established for negative integers n by observing that $e^{-ix} = \cos(-x) + i\ \sin(-x) = \cos\ x - i\ \sin\ x$ and $(e^{-ix})^n = \cos\ nx - i\ \sin\ nx$.

De Moivre's Theorem may be used to derive many trigonometric identities with a minimum of effort. For example,

$$\cos\ 2x + i\ \sin\ 2x = (\cos\ x + i\ \sin\ x)^2$$

implies

$$\cos\ 2x = \cos^2 x - \sin^2 x, \quad \sin\ 2x = 2\ \sin\ x\ \cos\ x.$$

Other identities of this type are given in Problem 11.32.

From the Euler Formula we also obtain

$$\cos\ x = \frac{1}{2}(e^{ix} + e^{-ix}), \quad \sin\ x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

and, since

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \tfrac{1}{2}(e^x - e^{-x}),$$

$$\cosh ix = \cos\ x, \qquad \sinh ix = i\ \sin\ x.$$

Finally, for any complex number $z = x + iy$, we may define

$$e^z = e^{x+iy} = e^x(\cos\ y + i\ \sin\ y) = e^x \cos\ y + ie^x\ \sin\ y.$$

We conclude this section with a discussion of the Simple Harmonic Oscillator and the Damped Harmonic Oscillator.

A point on the line whose position $x(t)$ at time t satisfies a differential equation

$$x''(t) + \omega^2 x(t) = 0 \qquad (\omega \neq 0)$$

is said to be in simple harmonic motion. For example, a mass m which moves subject to a force proportional to its distance from the origin and directed towards the origin satisfies an equation of the form $mx'' = -kx$ and so is in simple harmonic motion with $\omega = \sqrt{k/m}$. Fundamental facts about the differential equation are: (a) If $x_1(t)$ and $x_2(t)$ are solutions then so also is $x(t) = ax_1(t) + bx_2(t)$. (b) If $x_1(0) = x_2(0)$, $x_1'(0) = x_2'(0)$, then $x_1(t) = x_2(t)$ for all t (Problem 11.33). We may easily check that $x(t) = \cos \omega t$, $\sin \omega t$ are both solutions and that any solution x may be written in the form

$$x(t) = a \cos \omega t + b \sin \omega t$$

where $a = x(0)$, $b = x'(0)/\omega$. The solution x may also be written in the form

$x(t) = A \cos(\omega t - \varphi)$,

where (A, φ) are the polar coordinates of (a, b). This follows from

$x(t) = \sqrt{a^2 + b^2} \left[\dfrac{a}{\sqrt{a^2 + b^2}} \cos \omega t + \dfrac{b}{\sqrt{a^2 + b^2}} \sin \omega t \right]$

$= A[\cos \varphi \cos \omega t + \sin \varphi \sin \omega t]$.

Evidently all solutions are periodic with *period* $T = \frac{2\pi}{\omega}$ since $x(t + T) = x(t)$; the number A is called the *amplitude* and the angle φ is called the *phase* of the oscillation. The number of complete

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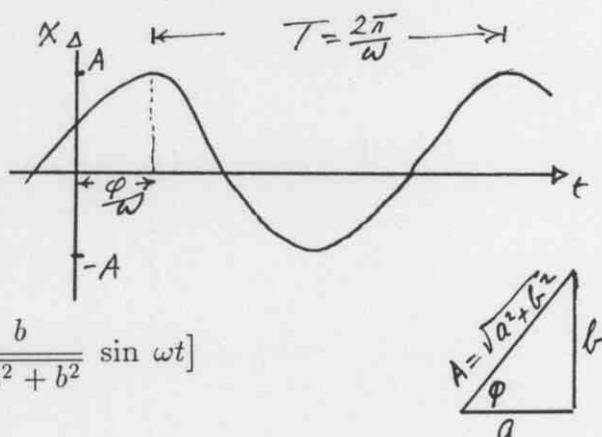
oscillations per unit of time $\frac{1}{T} = \frac{\omega}{2\pi}$ is called the *frequency*.

Finding the two solutions $\cos \omega t$, $\sin \omega t$ here involved a certain amount of guess work. The general approach for discovering solutions to equations of this type is to try to find a solution of the form $x = e^{\lambda t}$; then

$$x'' + \omega^2 x = (\lambda^2 + \omega^2)e^{\lambda t}$$

so that $e^{\lambda t}$ is a solution provided $\lambda^2 + \omega^2 = 0$; $\lambda = \pm i\omega$. But $e^{i\omega t}$, $e^{-i\omega t}$ are complex-valued. Two real-valued solutions are

$$\frac{1}{2}(e^{i\omega t} + e^{-i\omega t}) = \cos \omega t, \quad \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t}) = \sin \omega t.$$

If the motion of the oscillator is opposed by a force proportional to the velocity x', then $x(t)$ satisfies an equation of the form

$$x''(t) + 2\rho x'(t) + \omega^2 x(t) = 0, \quad \rho > 0.$$

If we try $x = e^{\lambda t}$, we find that

$$x'' + 2\rho x' + \omega^2 x = (\lambda^2 + 2\rho\lambda + \omega^2)e^{\lambda t}.$$

Thus $e^{\lambda t}$ is a solution provided $\lambda^2 + \rho\lambda + \omega^2 = 0$; $\lambda = \lambda_{1,2} = -\rho \pm \sqrt{\rho^2 - \omega^2}$. Solutions of the damped equation are of the form (Problem 11.33)

$$x(t) = ae^{\lambda_1 t} + be^{\lambda_2 t},$$
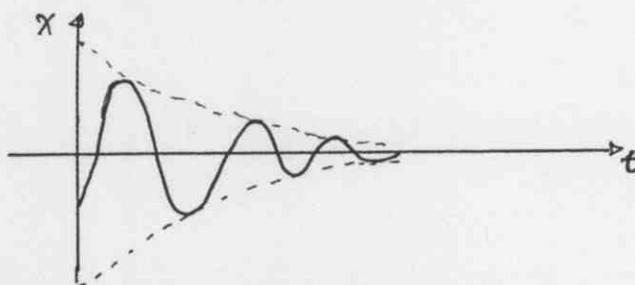
if $\rho^2 \neq \omega^2$.

Case (i) Damped Harmonic Oscillator $\rho < \omega$: In this case $\lambda = -\rho \pm i\omega_0$ where $\omega_0 = \sqrt{\omega^2 - \rho^2}$. All solutions are of the form

$$x(t) = ae^{(-\rho + i\omega_0)t} + be^{(-\rho - i\omega_0)t} = e^{-\rho t}(ae^{i\omega_0 t} + be^{-i\omega_0 t}).$$

In real form, the solutions may be written as

$$x(t) = e^{-\rho t}(a_1 \cos \omega_0 t + b_1 \sin \omega_0 t) = Ae^{-\rho t} \cos(\omega_0 t - \varphi).$$

Note that the effect of the damping is an oscillation whose amplitude decreases exponentially to zero in time and whose 'period' exceeds the undamped period, $T = \dfrac{2\pi}{\omega_0} > \dfrac{2\pi}{\omega}$. Note that the 'period' $\dfrac{2\pi}{\omega_0} \to \infty$ as $\rho \to \omega^-$.

Case (ii) Overdamping $\rho > \omega$: In this case λ_1 and λ_2 are both real and all solutions are of the form
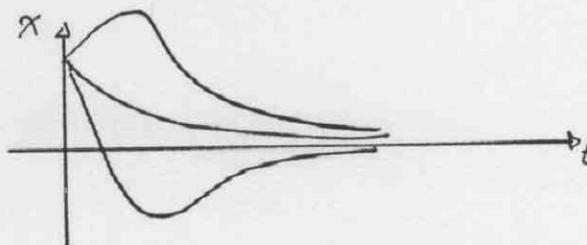
$x(t) = ae^{\lambda_1 t} + be^{\lambda_2 t}$

with $\lambda_1, \lambda_2 < 0$.

Here again all solutions $x(t) \to 0$, $t \to \infty$. But they do not oscillate and in fact each solution has at most one zero and at most one interior extremum.

Case (iii) Critical Damping $\rho = \omega$: Here solutions are of the form $x(t) = (a + bt)e^t$ (Problem 11.34), do not oscillate and satisfy $x(t) \to 0$, $t \to \infty$.

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§11.6. Improper Integrals

If f is continuous on $[a, \infty)$ we define the improper integral

$$\int_a^\infty f = \lim_{T \to \infty} \int_a^T f$$

if this limit exists. If the limit exists the integral is said to be *convergent*. Otherwise it is *divergent*.

EXAMPLE 11.6.1. $\int_0^\infty e^{-ax} dx = \frac{1}{a}$, if $a > 0$, and the integral is divergent, if $a \le 0$, since

$$\int_0^T e^{-ax} dx = \begin{cases} \frac{1}{a}(1 - e^{-aT}), & \text{if } a \ne 0 \\ T, & \text{if } a = 0. \end{cases}$$

EXAMPLE 11.6.2. $\int_1^\infty x^{-\alpha} dx = \frac{-1}{1-\alpha}$, $\alpha > 1$ and the integral is divergent, if $\alpha \le 1$, since

$$\int_1^T x^{-\alpha} dx = \begin{cases} \frac{1}{1-\alpha}(T^{1-\alpha} - 1), & \text{if } \alpha \ne 1 \\ \log T, & \text{if } \alpha = 1. \end{cases}$$

The *Cauchy Criterion* for the convergence of $\int_a^\infty f$ is that, for each $\varepsilon > 0$, there exists T such that

$$p, q \ge T \implies |\int_p^q f| < \varepsilon.$$

Since $\int_a^T f$ increases with T if f is positive $\int_0^T f$ tends to a limit as $T \to \infty$ if and only if it is bounded we obtain a comparison test similar to that for series.

PROPOSITION 11.6.3. *Suppose* f, g *are continuous and*

$$0 \le f(x) \le g(x), \quad a \le x < \infty.$$

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Then $\int_a^\infty g$ convergent $\implies \int_a^\infty f$ convergent.

We may also obtain the comparison test in a limit form.

PROPOSITION 11.6.4. *Suppose* f, g *are positive and continuous and satisfy*

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L.$$

(a) If $0 < L < \infty$, then $\int_a^\infty$ convergent $\iff \int_a^\infty f$ convergent.

(b) If $L = 0$, then $\int_a^\infty g$ convergent $\implies \int_a^\infty f$ convergent.

EXAMPLE 11.6.5 The integral $\int_1^\infty \frac{1}{x^{1/2}(x+1)} \, dx$ is convergent by the comparison text since $0 < \frac{1}{x^{1/2}(x+1)} < \frac{1}{x^{3/2}} = x^{-3/2}$, and $\frac{3}{2} > 1$.

EXAMPLE 11.6.6 $\int_1^\infty x^p e^{-x} dx$ is convergent for all p. Here we use the limit form of the Comparison Test with $f(x) = x^p e^{-x}$, $g(x) = e^{-\frac{1}{2}x}$. Thus

$$\frac{f(x)}{g(x)} = \frac{x^p e^{-x}}{e^{-\frac{1}{2}x}} = x^p e^{-\frac{1}{2}x} \xrightarrow[x \to \infty]{} 0, \quad \text{for all } p.$$

Proposition 11.6.4 (b) implies $\int_1^\infty f$ is convergent.

The improper integral $\int_{-\infty}^a f = \lim_{T \to \infty} \int_{-T}^a f$ may be handled similarly.

For functions f whose domain is a half-open interval $(a, b]$ or $[a, b)$ we may also consider the improper integrals

$$\int_{a+}^b f = \lim_{t \to a+} \int_t^b f, \quad \int_a^{b-} f = \lim_{t \to b-} \int_a^t f.$$
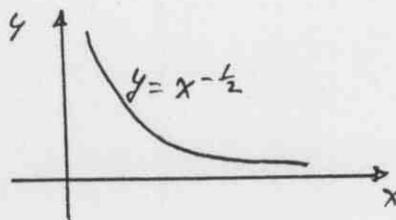
For functions f integrable on $[a, b]$, we have

$$\int_{a+}^{b} f = \int_{a}^{b} f \quad \text{and} \quad \int_{a}^{b-} f = \int_{a}^{b} f$$

but the improper integral may exist even if f is not integrable on $[a, b]$ in the Riemann sense.

EXAMPLE 11.6.7. $\int_{0+}^{1} x^p dx = \frac{1}{p+1}$, if $p > -1$ and is divergent if $p \leq -1$.

Thus $\int_{0+}^{1} x^{-1/2} dx = 2$ even though the integrand is unbounded on $(0, 1]$.

This follows from

$$\int_{t}^{1} x^p dx = \begin{cases} \frac{1}{p+1}(1 - t^{p+1}), & p \neq -1 \\ -\log t, & p = 1 \end{cases}$$

$$\xrightarrow[t \to 0+]{} \frac{1}{p+1}, \quad \text{if} \quad p > -1, \quad \text{and diverges otherwise.}$$

The various ideas of improper integral may be combined so that an integral may be improper at two ends:

$$\int_{0+}^{\infty} e^{-x} x^p dt = \int_{0+}^{1} e^{-x} x^p dx + \int_{1}^{\infty} e^{-x} x^p dx.$$

The first integral on the right converges, if $p > -1$, since $0 < e^{-x} x^p < x^p$, if $0 < x \leq 1$, and $\int_{0+}^{1} x^p dx$ is convergent (Example 11.6.7). The second integral is convergent for all p (Example 11.6.6).

In fact $\int_{0}^{\infty} e^{-x} x^p dx = p!$, if $p = 0, 1, \ldots$ (Problem 11.25(b)) and the integral may be taken as an extension to noninteger values of p of the factorial function.

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Problems

11.1. Determine which of the following series are convergent and which are divergent. Give reasons.

(a) $\sum_{k=1}^{\infty} \frac{k-2}{k+2}$,

(b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$,

(c) $\sum_{k=1}^{\infty} (\frac{3}{2})^k$,

(d) $\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)}$,

(e) $\sum_{k=1}^{\infty} (\frac{k+1}{k^2})$,

(f) $\sum_{k=1}^{\infty} \frac{1}{k!}$,

(g) $\sum_{k=2}^{\infty} \frac{1}{(\log\ n)^2}$,

(h) $\sum_{k=2}^{\infty} \frac{1}{n(\log\ n)^2}$,

(i) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$,

(j) $\sum_{k=1}^{\infty} \frac{(2n)!}{(n!)^2}$,

(k) $\sum_{j=0}^{\infty} \frac{2^j}{j!}$,

(l) $\sum_{j=1}^{\infty} \frac{2^j\,j!}{j^j}$,

(m) $\sum_{j=1}^{\infty} \frac{3^j\,j!}{j^j}$,

(n) $\sum_{k=0}^{\infty} e^{-k}$,

(o) $\sum_{k=1}^{\infty} \log(1+\frac{1}{k})$,

(p) $\sum_{k=1}^{\infty} \log(1+\frac{1}{k^2})$,

(q) $\sum_{k=1}^{\infty} \frac{\log\ k}{k^2}$,

(r) $\sum_{k=1}^{\infty} [\frac{1}{k} - \sin(\frac{1}{k})]$.

(s) $\sum_{k=1}^{\infty} \frac{1}{k} \sin(\frac{1}{k})$.

11.2. Which of the following are convergent? Absolutely convergent?

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$,

(b) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$,

(c) $\sum_{k=2}^{\infty} \frac{(-1)^k}{\log\ k}$,

(d) $\sum_{n=1}^{\infty} (-1)^n[\sqrt{n+1} - \sqrt{n}\,]$,

(e) $\sum_{n=1}^{\infty} \log(1+\frac{(-1)^n}{n})$,

(f) $\sum_{n=1}^{\infty} \frac{\sin\ n}{n^2}$.

11.3. Show that $\sum_{k=2}^{\infty} \frac{1}{(\log\ k)^p}$ is divergent for all $p \in \mathbb{R}$.

11.4. Show that $\sum_{k=2}^{\infty} \frac{(-1)^k}{k(\log\ k)^p}$ is convergent for all $p \in \mathbb{R}$ and is absolutely convergent if $p > 1$.

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11.5. Show that

(a) $\sum\limits_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}$,

(b) $\sum\limits_{n=1}^{\infty} \frac{2^n+3^n}{6^n} = \frac{3}{2}$,

(c) $\sum\limits_{k=1}^{\infty} \frac{k}{(k+1)!} = 1$,

(d) $\sum\limits_{k=1}^{\infty} \frac{1}{4k^2-1} = \frac{1}{2}$,

(e) $\sum\limits_{k=1}^{\infty} \frac{(k-1)!}{(k+p)!} = \frac{1}{p(p!)}$,

(f) $\sum\limits_{k=0}^{\infty} \frac{1}{(\alpha+k)(\alpha+k+1)} = \frac{1}{\alpha}$, $\alpha \neq 0, -1, -2, \ldots$

(g) $\sum\limits_{k=0}^{\infty} \frac{1}{(\alpha+k)(\alpha+k+1)(\alpha+k+2)}$ $= \frac{1}{2\alpha(\alpha+1)}$, $\alpha \neq 0, -1, \ldots$

11.6. Sum the series $\sum\limits_{k=0}^{\infty} \frac{1}{(\alpha+k)(\alpha+k+1)\ldots(\alpha+k+r)}$,   where r is a natural number.

11.7. Prove that $n^n e^{1-n} < n! < n^{n+1} e^{1-n}$ and that

$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}$.   [HINT: Consider $\int_1^n \log x \, dx$].

11.8. Let $u_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$.   Show tat $\{u_n\}$ is convergent and, if $\gamma = \lim_{n\to\infty} u_n$, $0 < \gamma < 1$.   N.B. γ is in the *open* interval $(0,1)$.

[HINT: Consider also the sequence

$v_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log(n+1)$ γ is called *Euler number*.

11.9. Assume the result of #11.8 and show that

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \log 2.$$

11.10. Assume the result of #11.8 and show that, if p, q are integers, $p > q$,   then

$$\lim_{n\to\infty} \left(\frac{1}{qn+1} + \frac{1}{qn+2} + \cdots + \frac{1}{pn} \right) = \log(\frac{p}{q}).$$

11.11. Assume the result of #11.8 and, by considering the sequence

$u_{2n} - u_n$, show that

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = \log 2.$$

11.12. If the terms in the series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ are rearranged by taking p positive terms followed by q negative terms, then the sum is

$$\log 2 + \frac{1}{2} \log(\frac{p}{q}).$$

11.13. Investigate the sequence

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n} \,.$$

11.14. We have seen that if $\sum_{k=0}^{\infty} a_k$ is convergent, then $\lim_{n\to\infty} a_n = 0$. Show that if $\{a_n\}$ is decreasing and $\sum_{k=0}^{\infty} a_k$ is convergent, then $\lim_{n\to\infty} na_n = 0$.

11.15. Prove that $\sum_{n=1}^{\infty} n^2 x^n = \frac{x^2+x}{(1-x)^3}$, if $|x| < 1$.

11.16. Prove that $\sum_{n=0}^{\infty} \frac{1}{(2n+1)n!} = \int_0^1 e^{x^2} dx$.

11.17. For what values of x is the series $\sum_{k=1}^{\infty} 2kx^{2k-1}$ convergent? What is its sum?

11.18. Prove that $\sum_{n=1}^{\infty} n(x - 2)^n$ is convergent if $1 < x < 3$. What is its sum?

11.19. Prove that $\int_0^x \frac{1}{1+t^8} \, dt = x - \frac{x^9}{9} + \frac{x^{17}}{17} - \frac{x^{25}}{25} + \dots$ if $|x| < 1$.

Show that equality also holds if $x = \pm 1$.

11.20. Prove that the power series $\sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdots (2k-1)}{2 \cdot 4 \cdots 2k} x^k$ is absolutely convergent if

$-1 < x < 1$, conditionally convergent if $x = -1$ and divergent if $x = 1$ or $|x| > 1$.

11.21. Show that $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$ is absolutely convergent if $|x| < e$ and divergent if $|x| \geq e$.

11.22. Investigate the series $\sum_{n=1}^{\infty} \frac{n^n}{n!} e^{-n}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!} e^{-n}$.

11.23. Use Taylor's Theorem to prove:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

for all x.

11.24. Prove that

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1 \cdot 2} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{1 \cdot 2 \cdot 3} x^3 + \dots$$

for all $\alpha \in \mathbb{R}$ if $|x| < 1$. When $\alpha = 0, 1, 2, \dots$, this is the Binomial Theorem and holds for all $x \in \mathbb{R}$.

HINT: Denote the series by $B(x)$. Show that $(1+x)B'(x) = \alpha B(x)$ and deduce $(1+x)^{-\alpha} B(x)$ is constant.

11.25. Show

(a) $\int_0^\infty \frac{dx}{a^2+x^2} = \frac{\pi}{2a}$, if $a > 0$.

(b) $\int_0^\infty t^n e^{-t} dt = n!$, $n = 0, 1, 2, \ldots$.

(c) $\int_1^\infty \frac{\log x}{x^2} dx = 1$.

(d) $\int_{0+}^1 \log x \, dx = -1$.

(e) $\int_0^\infty \frac{x^2 dx}{1+x^2+x^4} = \frac{\pi}{2\sqrt{3}}$.

(f) $\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$.

(g) $\int_0^{1-} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$.

(h) $\int_1^\infty \frac{dx}{x+x^{n+1}} = \frac{1}{n} \log 2$, $n > 1$.

11.26. Let $a_k > 0$ and $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = c$. Show that, if $c \neq 1$, the sequence is ultimately monotone and

(a) $\lim_{k\to\infty} a_k = 0$, if $0 \leq c < 1$

(b) $\lim_{k\to\infty} a_k = \infty$, if $c > 1$.

11.27. If $c = 1$ in the preceding example, the sequence may not be ultimately monotone. However if $\frac{a_{k+1}}{a_k} = 1 - \frac{p}{k} + o(\frac{1}{k})$, $k \to \infty$, the sequence is ultimately monotone if $p \neq 0$ and satisfies $\lim_{k\to\infty} a_k = 0, \infty$ as $p > 0, < 0$ respectively. Prove this, the assertion of Lemma 11.3.6 (Raabe's Test for Sequences).

11.28. Prove that $\sum_{k=1}^\infty (-1)^k [\frac{1\cdot 3\cdot 5 \ldots (2k-1)}{2\cdot 4\cdot 6 \ldots 2k}]^p$ is absolutely convergent if $p > 2$, conditionally convergent if $0 < p \leq 2$ and divergent if $p \leq 0$.

11.29. (Root Test) Suppose $a_n \geq 0$ and

$$\lim_{n \to \infty} a_n^{1/n} = c.$$

Show that

(a) if $0 \leq c < 1$, then $\sum_{k=1}^{\infty} a_k$ is convergent

(b) if $c > 1$, then $\sum_{k=1}^{\infty} a_k$ is divergent

(c) if $c = 1$, then the series may be convergent or divergent.

11.30. (Cauchy Condensation Test)

(a) Suppose $a_n > 0$ is decreasing. Show that $\sum_{k=1}^{\infty} a_k$ convergent $\iff \sum_{k=1}^{\infty} 2^k a_{2^k}$ convergent.

(b) Use (a) to show $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

(c) Use (a) to investigate the series

$$\sum_{k=2}^{\infty} \frac{1}{k(\log k)^p}, \quad \sum_{k=2}^{\infty} \frac{1}{k \log k (\log \log k)^p}.$$

11.31. Use de Moivre's Theorem to show that

$$(\cos x + i \sin x)^{m/n} = \cos(\frac{m}{n} x) + i \sin(\frac{m}{n} x).$$

First decide what the left-hand side of this expression should mean.

11.32. Show that

$$\cos 3x = 4 \cos^3 x - 3 \cos x, \qquad \sin 3x = 3 \sin x - 4 \sin^3 x,$$

$$\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1, \qquad \sin 4x = 8 \cos^3 x \sin x - 4 \cos x \sin x.$$

11.33. (a) Suppose $x = x_1(t)$, $x = x_2(t)$ both satisfy the equation

$$x'' + 2\rho x' + \omega^2 x = 0, \ x(0) = a, \ x'(0) = b \quad (\rho \geq 0).$$

Show that $x_1(t) = x_2(t)$ for all $t \geq 0$.

HINT: Show $g(t)$ decreases, where $g(t) = \left(x_1'(t) - x_2'(t)\right)^2 + \omega^2 \left(x_1(t) - x_2(t)\right)^2$.

(b) Suppose $x = x_1(t)$, $x = x_2(t)$ are as in (a) and neither is a constant multiple of the other. Show that all solutions of $x'' + \rho x' + \omega^2 x = 0$ are of the form $x(t) = ax_1(t) + bx_2(t)$.

11.34. If $\rho = \omega$, show that all solutions of $x'' + 2\rho x' + \omega^2 x = 0$ are of the form $x(t) = (a + bt)e^{-\rho t}$.

11.35. Prove the formulas

$$1 + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin \frac{1}{2} x + \sin(n + \frac{1}{2})x}{2 \sin \frac{1}{2} x},$$

$$\sin x + \sin 2x + \cdots + \sin nx = \frac{\cos \frac{1}{2} x - \cos(n + \frac{1}{2})x}{2 \sin \frac{1}{2} x}.$$

HINT: If $z \neq 1$ is any complex number, show

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

and

$$\frac{1 - e^{i(n+1)x}}{1 - e^{ix}} = \frac{e^{-\frac{i}{2} x} - e^{i(n+\frac{1}{2}) x}}{e^{-\frac{i}{2} x} - e^{\frac{i}{2} x}}.$$