Iverson computing competition
2015 may 26

name ____________________________________

school ____________________________________
city __________________
grade _________
cs teacher ____________________________________

are you taking AP or IB computer science? (yes/no) _______________________
have you taken advanced level courses? (3000 level, e.g. CSE3110 iterative algorithms I.) (yes/no/currently taking) __________________________________

illegible answers will not be marked

question	– – – –	marks	your score
1	continents	10	
2	permutations	10	
3	pebble game	10	
4	fractions	3	
total	– – – –	33	

Exam Format

This is a two-hour paper and pencil exam. There are four questions, each with at least one part. Some part(s) might be easy. Solve as many parts of as many questions as you can.

Programming Language

Questions that require programming can be answered using any language (e.g. C/C++, Java, Python, ...) or pseudo-code. **Pseudo-code should be detailed enough to allow for a near direct translation into a programming language.** Clarify your code with appropriate comments. For full marks, a question must be correct, well-explained, and as simple as possible.

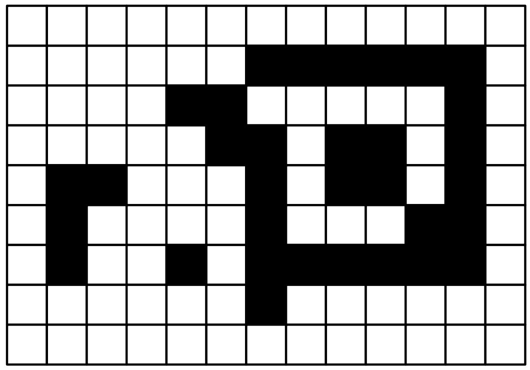
Our primary interest is in thinking skill rather than coding wizardry, so logical thinking and systematic problem solving count for more than programming language knowledge.

Suggestions

1. You can assume that the user enters only valid input in the coding questions.
2. In somes cases, sample executions of the desired program are shown. Review the samples carefully to make sure you understand the specifications. The samples may give hints.
3. Design your algorithm before writing any code. Use any format (pseudo-code, diagrams, tables) or aid to assist your design plan. We may give part marks for legible rough work, especially if your final answer is lacking. We are looking for key computing ideas, not specific coding details, so you can invent your own "built-in" functions for subtasks such as reading the next number, or the next character in a string, or loading an array. Make sure to specify such functions by giving a relationship between their inputs and outputs.
4. Read all questions before deciding which ones to attempt, and in which order. Start with the easiest parts of each question.

question 1: continents

Pixel-world is a peculiar planet. It is a flat world that is divided into a grid of square cells. Each cells is completely covered by land or completely covered by water. Example: in the grid below, the dark cells represent land and the light cells represent water.

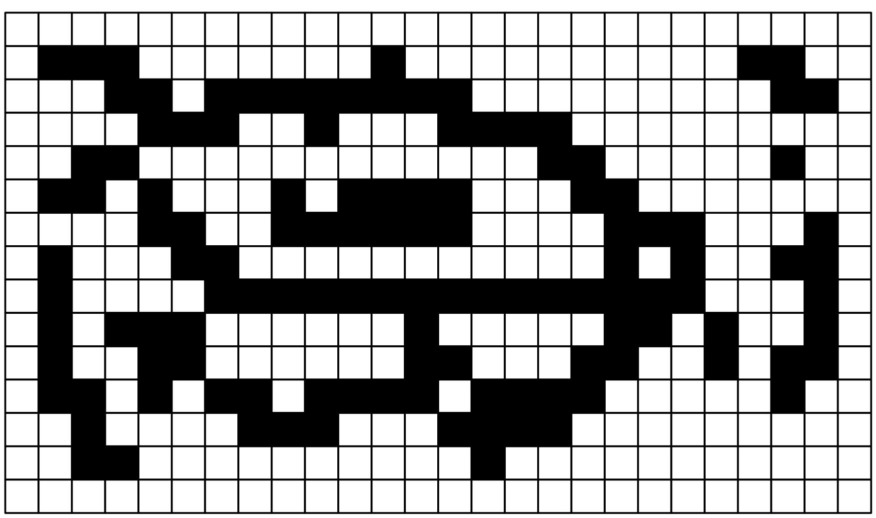


A continent is a collection of land cells, say C, such that

- it is possible to walk between any two cells in C by taking horizontal or vertical steps (no diagonal steps) and never entering a water cell.
- every land cell that can be reached in this way from a land cell in C is also in C.

The size of the continent C is the number of cells in C. Example: the grid shown above has 4 continents, with sizes 1, 4, 4, 24.

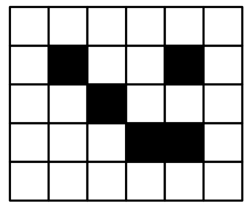
a) [2 marks] Give the number of continents in the following grid.



Solution: There are 9 continents.

b) [2 marks] Suppose, for each land cell, we already know the size of the continent that includes that cell. These sizes are stored in an array and the array is sorted.

Example: `1, 1, 1, 2, 2` is the sorted list of these sizes for the following grid. The size `2` appears twice because it is entered into the array once for each land cell in the continent.



The list below was obtained in this way from a grid with 30 land cells. How many continents does this grid have? Explain briefly.

```
1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3,
3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4
```

Solution: A continent of size k will be reported k times in the list. For $i \geq 1$, if we let a_i denote the number of times i appears in the list then the answer is

$$\frac{a_1}{1} + \frac{a_2}{2} + \frac{a_3}{3} + \dots$$

In this case, the number of continents is

$$\frac{2}{1} + \frac{4}{2} + \frac{12}{3} + \frac{12}{4} = 11.$$

c) [3 marks] Write a function `count_continents(sizes, n)` where `sizes` is a sorted array of `n` integers, obtained from a grid with `n` land cells. The function should return the number of continents in the grid.

Solution: Here is a Python implementation that follows the formula from the previous part.

```
def count_continents(sizes, n):
    tot = 0
    for i in range(1,n):
        tot += sizes.count(i)//i
    return tot
```

It is a bit slow for large inputs because the `count()` method will scan `sizes` every iteration of the loop. The following is faster because it only walks through the list once. It takes advantage of the fact that all occurrences of a number will appear consecutively (because the list is sorted).

```
def count_continents(sizes, n):
    prev = 0
    tot = 0

    for j in range(1,n):
        if sizes[j] != sizes[prev]:
            tot += (j-prev)//sizes[prev]
            prev = j

    tot += (n-prev)//sizes[prev]
    return tot
```

d) [3 marks] We represent a grid by specifying two positive integers, `rows` and `columns` — indicating the grid size — together with a two-dimensional array `grid`. For $1 \leq$ `r` $\leq$ `rows` and $1 \leq$ `c` $\leq$ `columns`, `grid[r][c]` is `0` if the cell at location `(r, c)` is water, and is `1` if that cell is land.

Write a function `continent_size(r, c, rows, columns, grid)` that returns the size of the continent that includes the cell at location `(r, c)`. If this is a water cell, return `0`. You may assume that every cell on the boundary of the grid is water.

Solution: Mark off the cells in the continent one at a time. Initially, mark the cell at coordinate `(r,c)`. While there is marked land cell that is adjacent to an unmarked land cell, then mark that unmarked cell. When there are no more cells to mark, return the number of marked cells.

```
#return the list of the four cells adjacent to cell (r, c)
def neighbours(r, c):
    return [(r-1, c), (r+1, c), (r, c-1), (r, c+1)]

def continent_size(r, c, rows, columns, grid):
    if grid[r][c] == 0:
        return 0

    #create a grid of 0s
    marked = [[0]*columns]*rows
    marked[r-1][c-1] = 1

    count = 1


    while true:
        found = false

        for cr in range(rows):
            for cc in range(columns):
                if marked[cr][cc]:

                    #examine the neighbours of the marked cell
                    for (nr, nc) in neighbours(cr, cc):

                        #mark the neighbour if is is an unmarked land cell
                        if not marked[nr][nc] and grid[nr][nc]:
                            marked[nr][nc] = 1
                            count += 1
                            found = true

        if not found:
            # the entire continent must be marked if we reach here
            return count
```

This can be made more efficient. Notice that we only have to examine the neighbours of a marked cell once.

The following Python code exploits this fact. It maintains a list `to_examine` that contains the cells that have been marked but have not yet had their neighbouring cells checked. It removes one cell from this list at a time and checks the neighbours of that cell. If any neighbour is a land cell that is not yet marked, it is marked and then added to the list. This way, every cell the continent has its neighbours checked exactly once.

```
#return the list of the four cells adjacent to cell (r, c)
def neighbours(r, c):
    return [(r-1, c), (r+1, c), (r, c-1), (r, c+1)]

def continent_size(r, c, rows, columns, grid):
    if grid[r][c] == 0:
        return 0

    #create a grid filled with 0s
    marked = [[0]*columns]*rows
    marked[r-1][c-1] = 1

    #the list of cells that have been marked
    to_examine = [(r-1, c-1)]

    #the number of cells in the continent we have marked so far
    count = 1

    while len(to_examine) > 0:
        #remove something from the list of unprocessed tiles
        (cr, cc) = to_examine.pop()

        #examine the neighbours of this cell
        for (nr, nc) in neighbours(cr, cc):
            #if the neighbouring tile is an unmarked land tile
            #then mark it and add it to the list of tiles to process
            if marked[nr][nc] == 0 and grid[nr][nc] == 1:
                marked[nr][nc] = 1
                count += 1
                to_examine.append((nr, nc))

    return count
```

question 2: divisibility testing

In this question, we describe numbers in both decimal and binary form. A decimal number is subscripted with 10; a binary number is subscripted with 2. Example: $5_{10} = 101_2$.

You might know some divisibility tests for decimal numbers. Example: a number is divisible by 3 if and only if the sum of its decimal digits is divisible by 3. This rule does not hold for binary digits: $\mathtt{3}_{10} = \mathtt{11}_2$ but the sum of its binary digits is $\mathtt{2}_{10}$.

We will use *finite state machines* (FSMs) for our tests. A FSM is a computational device that reads in a string of bits (`0` or `1`) and decides whether to accept that string. It has these features:

- **states**, which are depicted as labelled circles (`a, b, c` in the example below);
- for each state `x`, there are precisely two **arcs** that start at `x` and point to some other state (perhaps `x` again); one arc is labelled `0` and the other `1`;
- one state is the **start** state and one is the **accepting** state; the start state has an arrow pointing to it labelled `start`; the accepting state has a thick border; the start and accepting states can be the same.

A computation with a FSM is simple to describe. A "current state" v is maintained which is initialized to be the start state. The input string is read one bit at a time, from left to right. When a bit `b` is read, the current state v is updated to be the state that is pointed to from v by the arc labelled `b`.

Once the entire input is processed, the FSM *accepts* the string if v is the accepting state, otherwise it *rejects* the string. We assume that the input string has at least one bit.

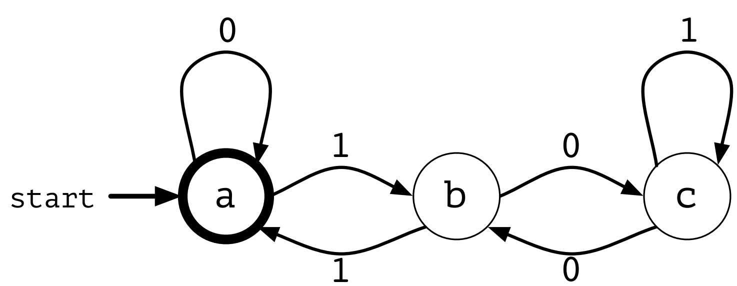


Figure 1: A finite state machine that accepts binary numbers divisible by 3. Here, `a` is both the start state and the accepting state.

Given a FSM, we can illustrate a computation by writing the sequence of states assumed by v, connecting consecutive labels with an arrow that labelled by the associated input bit. Example: for the FSM above, here is the computation for input string `110`:

$$\mathtt{a} \xrightarrow{1} \mathtt{b} \xrightarrow{1} \mathtt{a} \xrightarrow{0} \mathtt{a}$$

and here is the computation for input string `100`:

$$\mathtt{a} \xrightarrow{1} \mathtt{b} \xrightarrow{0} \mathtt{c} \xrightarrow{0} \mathtt{b}$$

String `110` is accepted because the final state is the accepting state, but string `100` is rejected. In fact, this FSM accepts precisely the binary numbers that are divisible by $3_{10} = 11_2$. Notice that both `011` and `00` are accepted; leading 0s are allowed.

a) [2 marks] Illustrate the computation of the above FSM for (i) input string `1010010` and (ii) input string `1011010`. For each string, state whether it is accepted.

Solution:

$$\mathtt{a} \xrightarrow{1} \mathtt{b} \xrightarrow{0} \mathtt{c} \xrightarrow{1} \mathtt{c} \xrightarrow{0} \mathtt{b} \xrightarrow{0} \mathtt{c} \xrightarrow{1} \mathtt{c} \xrightarrow{0} \mathtt{b}$$

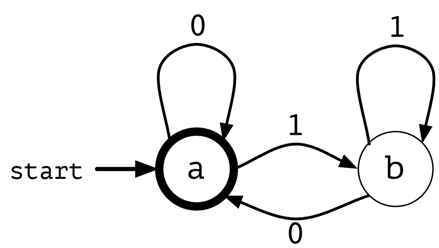
The string `1010010` is rejected. Note: $1010010_2 = 82_{10}$ which is not divisible by 3.

$$\mathtt{a} \xrightarrow{1} \mathtt{b} \xrightarrow{0} \mathtt{c} \xrightarrow{1} \mathtt{c} \xrightarrow{1} \mathtt{c} \xrightarrow{0} \mathtt{b} \xrightarrow{1} \mathtt{a} \xrightarrow{0} \mathtt{a}$$

The string `1011010` is rejected. Note: $1011010_2 = 90_{10}$ which is divisible by 3.

b) [2 marks] Draw a FSM that accepts precisely the binary numbers that are divisible by 2_{10} (leading zeros are allowed).

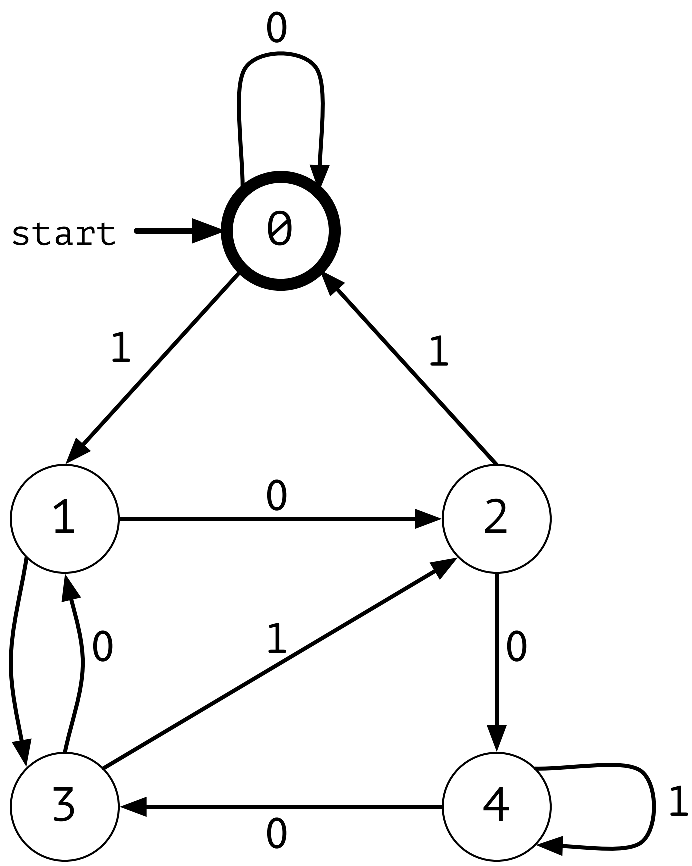
Solution: Even numbers are precisely those whose binary representation ends in a `0`.



c) [3 marks] Draw a FSM that accepts precisely the binary numbers that are divisible by 5_{10} (leading zeros are allowed).

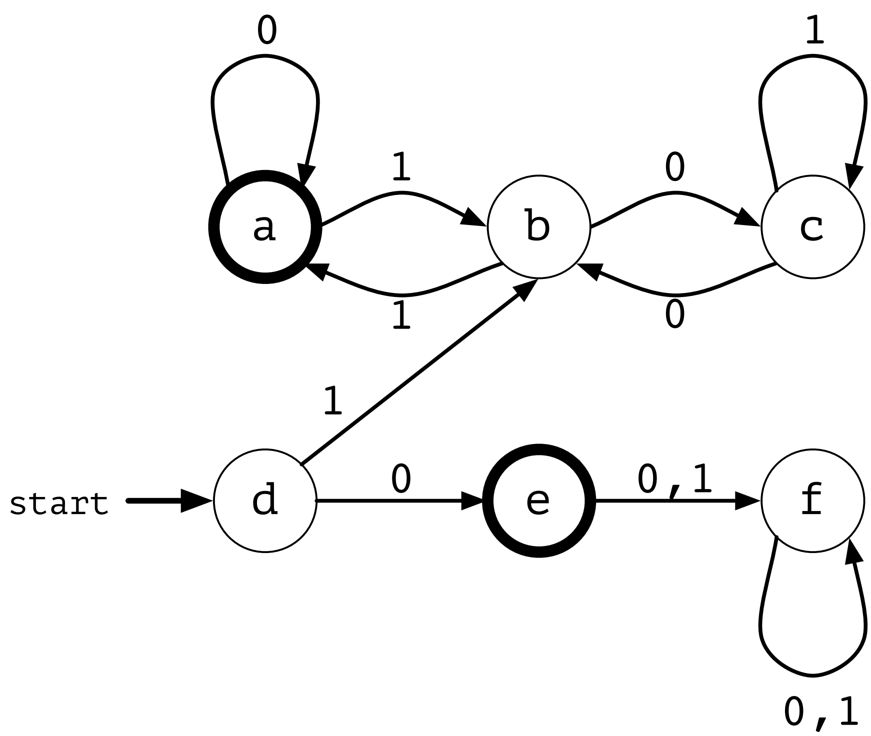
Solution: Consider how the binary number changes as the bits are read in. If a `0` is read then the number is multiplied by 2. If a `1` is read then the number is multiplied by 2 and increased by 1. Example: `101` represents 5 and `1011` represents $2 \cdot 5 + 1 = 11$.

The states of the FSM will keep track of the value *mod 5* of the binary number read so far and the arcs model how this value changes as the bits are read. Example: consider a binary number that is 3 *mod 5*. Suppose `1` is then appended to the number. The value *mod 5* of the new number then becomes $2 \cdot 3 + 1 \equiv 2$.



d) [3 marks] Draw a FSM from the description that accepts precisely binary numbers that are divisible by 3_{10} and either are the number zero or have a leading 1. Example: `11` and `0` are accepted; `011`, `00`, `10` are rejected.

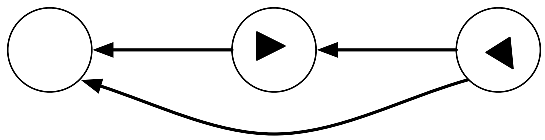
Solution: The only allowed string that starts with `0` is `0` itself. The following FSM will move into the "divisibility by 3" FSM if a leading `1` is read. If a leading `0` is read, the rest of the FSM (states `e,f` below) makes sure nothing else is read.



question 3: pebble game

Alice and Bob play a 2-player game. A number of pebbles are placed in various *positions* that are arranged horizontally. On a *turn*, a player moves one pebble and to the left. However, not all such moves are valid; valid moves are specified as part of the game. If no pebble can be moved, then the player whose turn it is loses and the other player wins.

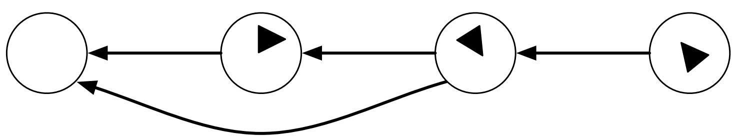
Example: here is a game with 3 positions (circles) and 2 pebbles (triangles). The arcs show the valid moves.



Here, Alice can win the game by moving the rightmost pebble to the middle. Now Bob's only option is to move one of these pebbles to the leftmost point; then Alice moves the other pebble left, and Bob has no moves so Alice wins.

In the following questions, assume both Alice and Bob play perfectly. That is, if the current player can move so that that they can win by continuing to play perfectly, then they make a winning move.

a) [2 marks] Who wins this game? Remember, Alice plays first. Explain briefly. It might help to label the pebbles.



Solution: Call the pebbles a, b, c in order from left to right.

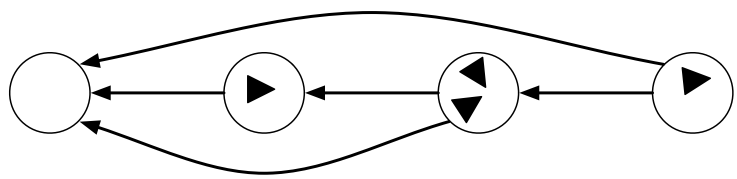
Alice can force a win. She moves pebble b to the position with pebble a. She can now guarantee a win through the given strategy:

- When Bob moves pebble c, Alice responds by moving c to the leftmost position.
- When Bob moves one of a or b to the leftmost position, Alice responds by moving the other to this position.

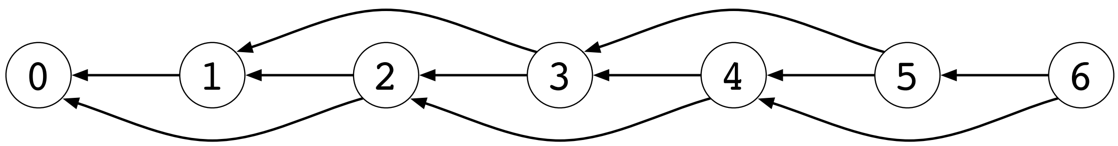
b) [2 marks] Who wins this game? Explain briefly.

Solution: Label the pebbles a, b, c, d in left to right order.
Bob will win through the following strategy.

- When Alice moves either a or d to the leftmost position, Bob responds by moving the other to the leftmost position.
- When Alice moves either b or c, Bob responds by moving the other to the same position.



c) [3 marks] Consider the game with n positions numbered from 0 to $n-1$ (left to right) and only these valid moves: for $0 \leq v \leq n-1$, a pebble can move from v to $v-1$; for $v \geq 2$, a pebble can move from v to $v-2$. The game with $n = 7$ is shown below.



Write a function `winner(i, j)` that determines who wins on such a board with one pebble at location $i > 0$ and one pebble at location $j > 0$, where possibly $i = j$. Return a string, either `"Alice"` or `"Bob"`.

Explain briefly why your code is correct. For full marks, your algorithm should run in a fraction of a second, even for i and j as large as 10^9. *Hint*: there is a nice pattern.

Solution: Alice wins if `i` and `j` are different *mod 3*, Bob wins if they are the same *mod 3*. A short explanation is that if they are different *mod 3* then Alice is able to make them the same by subtracting 1 or 2 from the larger *mod 3* value. If this did not end the game, then Bob's only move results in them being different *mod 3* and Alice repeats. If they are initially the same *mod 3*, then Bob just follow's Alice's strategy described above.

A more precise explanation follows. Change the label of each position i to i *mod 3*. In left-to-right order, the labels are $0, 1, 2, 0, 1, 2, 0, 1, 2, \ldots$.

Let us call the placement of two pebbles *similar* if they lie on positions with the same label. Otherwise they are *dissimilar*. Note that the end placement (both pebbles on 0) is similar.

- There is a way to turn any dissimilar placement into a similar placement by moving one pebble. Move the pebble on the higher *mod 3* label to match the other.
- Any single move from a similar placement will result in a dissimilar placement because there is no move between positions with the same label.

Whenever a player plays from a dissimilar placement, then make the move that results in a similar placement. If this did not finish the game, then the other player is forced to play from a similar position which creates a dissimilar position (and cannot be the last move in the game). This strategy is repeated. Thus, if the initial placement is dissimilar then Alice will win, otherwise Bob will win.

```
def winner(i, j):
    if i % 3 == j % 3:
        return "Bob":
    else:
        return "Alice"
```

d) [3 marks] Now consider the same board as the previous part, except we may have many pebbles on the board. Write pseudocode for a function `win(a, m, n)` where `n` is the number of positions on the board and `a[]` is an array with `m` nonnegative integers, each between 0 and `n-1`, specifying the initial placement of the pebbles. Again, this code should run quickly and you must explain briefly why it is correct.

Solution:
Call a placement of pebbles *similar* if both a_1 and a_2 are even, and *dissimilar* otherwise. Note the game ends in a similar placement.

- If the placement is dissimilar, then there is a single move that makes it similar. That is, if exactly one of a_1 or a_2 is odd then move a single pebble of the corresponding label to a position with label 0. If both a_1 and a_2 are odd then move a pebble from a position with label 2 one a position with label 1.
- If the placement is similar, then every move makes it dissimilar. If the move is from either a label 1 or a label 2 position, then the corresponding value a_1 or a_2 becomes odd. If the move is from a label 0 position, then it ends on a label 1 or label 2 position making the corresponding a_1 or a_2 odd.

Alice wins if the initial placement is dissimilar and Bob wins if the initial placement is similar by following essentially the same strategy as in part c, except using this notion of similar and dissimilar.

```
def win(a, m, n):
    a1 = 0
    a2 = 0

    for i in a:
        if i%3 == 1:
            a1 += 1
        elif i%3 == 2:
            a2 += 2

    if a1%2 == 0 and a2%2 == 0:
        return "Bob"
    else:
        return "Alice"
```

question 4: fractions

You are given a fraction $\frac{a}{b} \geq 0$. However, a and b might be large. So, given a positive integer N, you want to find a "simpler" fraction $\frac{c}{d}$ that is as close to $\frac{a}{b}$ as possible but with $0 \leq c, d \leq N$. That is, you should find c and d so that $0 \leq c \leq N$, $0 < d \leq N$, and $\left|\frac{a}{b} - \frac{c}{d}\right|$ as small as possible. If there are multiple solutions, choose the answer such that $c + d$ is as small as possible.

[3 marks] Write a function `simpler(a,b,N)` that prints `closest simpler fraction c / d`

where `c` and `d` are the numerator and denominator of the answer. Remember, you can write helper functions.

Example: calling `simpler(5, 7, 5)` prints `closest simpler fraction 3 / 4`

and calling `simpler(11, 17, 7)` prints `closest simpler fraction 2 / 3`

Solution: Simply iterate over all pairs (c, d) with $0 \leq c \leq N$ and $1 \leq d \leq N$. If one of them forms a fraction that is closer than the previous best, then keep it.

The following code does exactly this. Note, that it never explicitly checks that $c + d$ is the smallest among all possible answers. The order we iterate over c, d guarantees the first time we encounter a closest fraction that it will have minimum $c + d$ value. Can you see why?

```
# is the fraction num/den closer to a/b than c/d?
def closer(num, den, c, d, a, b):
    #num1/den1 = distance between num/den and a/b
    num1 = abs(num*b - a*den)
    den1 = den*b

    #num2/den2 = distance between a/b and c/d
    num2 = abs(c*b - a*d)
    den2 = d*b

    #cross multiply to check num1/den1 < num2/den2
    return num1*den2 < num2*den1

def simpler(a, b, N):
    c = 0
    d = 1

    for num in range(1,N+1):
        for den in range(1,N+1):
            if closer(num, den, c, d, a, b):
                c, d = num, den

    print("closest simpler fraction", c, "/", d)
```