Lesson 1: The SIS Model of Disease Transmission

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This is the final step before the students can begin working on the West Nile model. What we would like to accomplish here is to show how a simple model of disease transmission can be developed from a basic understanding of the derivative, and show how it relates to the logistic equation.

Imagine a closed population, where individuals neither leave nor arrive and the birth and death rates are zero. This is, roughly, what a school might be like over the course of a single semester. Let *N* be the size of this population.

Now suppose that one person in this population is infected with an infections non-fatal disease, such as a cold. We wish to describe the spread of the disease through the population. To do this, we divide the population into two groups:

Susceptibles: These are the healthy individuals.

Infectives: These are the people who currently carrying the disease and are capable of infecting the susceptibles.

Let *S(t)* be the number of susceptibles in the population at time *t*.

Let *I(t)* be the number of infectives in the population at time *t*.

The model is called **SIS** because susceptibles become infectives then recover and become susceptibles again.

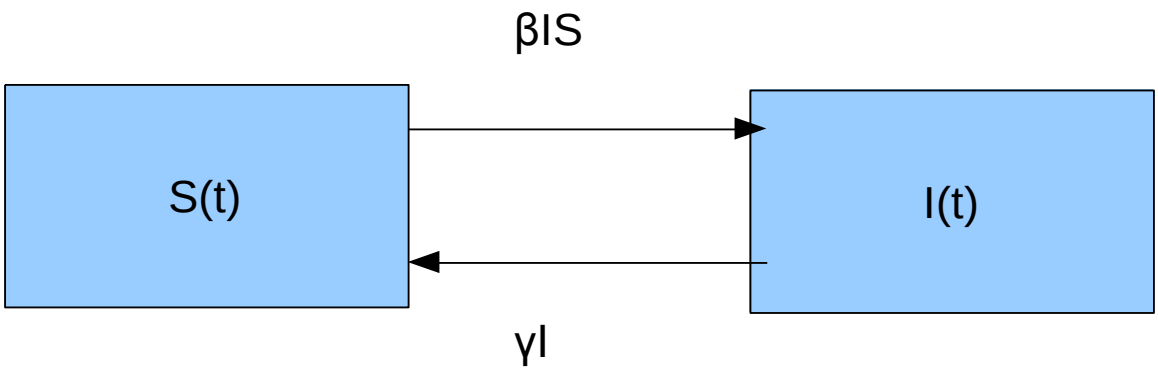
$S \to I \to S$

By our original assumption that the population is closed, we have $N = S(t) + I(t)$.

Questions:

1. What can we say about the signs of *S(t)* and *I(t)*?
2. How would we interpret the signs of $\frac{dS}{dt}$ and $\frac{dI}{dt}$?

We can make a diagram of the changes in *S* and *I*. The boxes represent variables and the arrows show population changes as the product of variables and parameters.



The diagram indicates that infectives recover at a constant per-capita rate of γ, so that at any given time, the number of infectives that recover (and become susceptible again) is γI.

To become infected, a susceptible must come into contact with an infective. By the fundamental counting principle, the number of interactions between *S* and *I* is proportional to their product, assuming that individuals are randomly interacting. So we get the number of infections at any given time given by βIS. We call β the *infectivity of the disease*.

Question: Consider two diseases. One has β=0.1 and the other has β=0.05. In biological terms, what is the difference between the two diseases?

Answer: The disease with β=0.1 is more contagious.

Question: Suppose a drug is given to infectives so that they recover more quickly than they otherwise would. What effect would this have on γ?

Answer: If γ is larger, then infectives that are recovering in greater numbers than the same sized pool would recover otherwise. It must be the case that they are recovering more quickly. Therefore, the drug increases the value of γ.

Now, let's translate the diagram into equations.

$\frac{dS}{dt} = -\beta IS + \gamma I$ because the rate of change of susceptibles is the per-capita rate of healing minus the per-capita rate of infection.

$\frac{dI}{dt} = \beta IS - \gamma I$

We see that the rate of change of susceptibles is the negative of the rate of change of the infectives.

Question: Why?

Answer: Recall that *N*, the total population, is assumed to be constant.

$$N = S + I$$
$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt}$$
$$0 = \frac{dS}{dt} + \frac{dI}{dt}$$
$$\frac{dS}{dt} = -\frac{dI}{dt}$$

We will now reduce the two differential equation to one, by using the fact that *N*=*S*+*I*.

$$\frac{dI}{dt} = \beta IS - \gamma I$$
$$\frac{dI}{dt} = \beta I(N - I) - \gamma I$$
$$\frac{dI}{dt} = \beta IN - \beta I^2 - \gamma I$$
$$\frac{dI}{dt} = (\beta N - \gamma)I - \beta I^2$$

Question: Do you recognize this equation?

Answer: It is not obvious, but it is the logistic equation.

We can rearrange the equation as follows.

$$\frac{dI}{dt} = (\beta N - \gamma)I - \beta I^2$$
$$\frac{dI}{dt} = (\beta N - \gamma - \beta I)I$$
$$\frac{dI}{dt} = (\beta N - \gamma)\left[1 - \frac{I}{\frac{\beta N - \gamma}{\beta}}\right]I$$

This looks worse than it is. Make the substitutions of $r = \beta N - \gamma$ and $K = \frac{\beta N - \gamma}{\beta}$, then we simply get

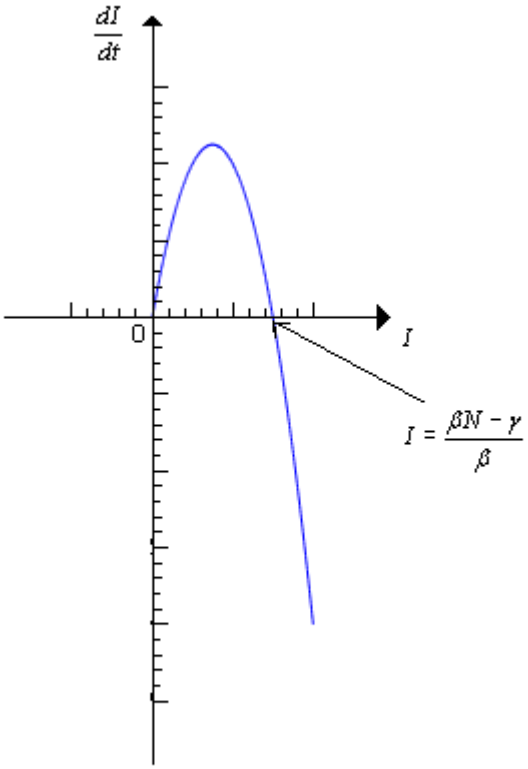
$\frac{dI}{dt} = r\left(1 - \frac{I}{K}\right)I$, which is the logistic equation.

Interpretation.

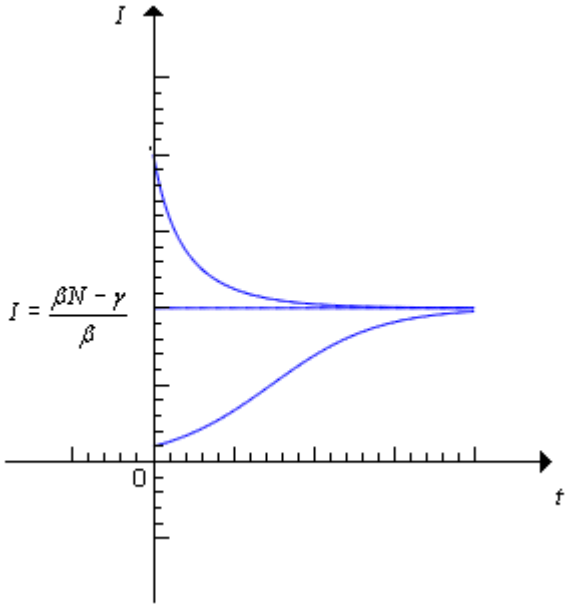
The question now is: What does this mean in a biological system? Consider two cases separately. Look at the phase-line plots when $\beta N - \gamma > 0$ and when $\beta N - \gamma < 0$.

Case 1: $\beta N - \gamma > 0$

This case looks exactly like the solution to the logistic equation in the previous lesson.



As we saw before, this has the solution:

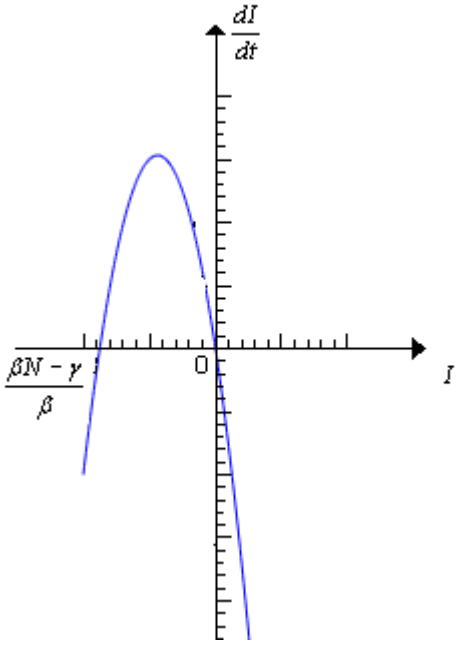


The conclusion is that when $\beta N - \gamma > 0$, there will always be $\frac{\beta N - \gamma}{\beta} = N - \frac{\gamma}{\beta}$ infectives in the population.

Recall that β is the rate of infection and γ is the rate of recovery. It turns out that the number of people infected depends on their ratio.

Case 2: $\beta N - \gamma < 0$

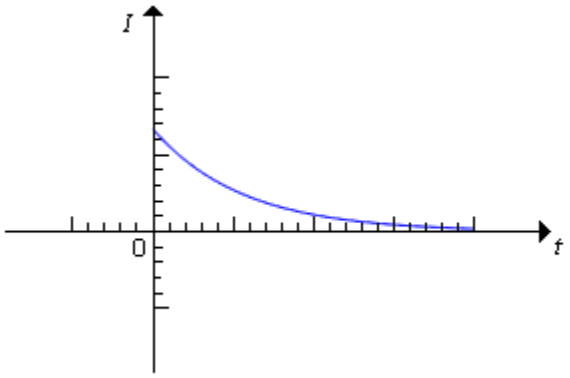
This results in Case 1's phase-line graph being reflected in the vertical axis:



In this phase-line graph we notice two things. First, in the biological application $I \geq 0$ because it counts the number of members of the population that are infected.

Second, for the part of the domain that applies, I is always decreasing because $\frac{dI}{dt} < 0$.

The time-domain graph, then, looks something like this.



In this case, the number of infectives approaches zero. That is, the disease will eventually be eliminated from the population.

Question: In light of these graphs, what strategies might be adopted to eliminate the disease from the population?

Answer: Just as we would expect, anything that either slows down the rate of transmission, such as vaccination or bacterial control would help, as would medications or practices that speed up the rate of recovery. What is critical is that their ratio change: once $\frac{\gamma}{\beta} > N$ the disease will be eliminated.

Conclusion (and introduction to R_0)

From the phase line analysis, we found that the disease is *endemic* (it persists in the population) if $\beta N - \gamma > 0$, and that the disease is eliminated from the population if $\beta N - \gamma < 0$.

We can rewrite the inequalities as follows:

$$\beta N - \gamma > 0$$
$$\Leftrightarrow \beta N > \gamma$$
$$\Leftrightarrow \frac{\beta N}{\gamma} > 1$$

and, of course

$$\beta N - \gamma < 0 \Leftrightarrow \frac{\beta N}{\gamma} < 1.$$

This allows us to note the factor that is crucial to the spread or control of the disease.

Let R_0 be called the *basic reproduction ratio of the disease.*

$$R_0 = \frac{\beta N}{\gamma}$$

If $R_0 > 0$, then the disease is endemic to the population.

If $R_0 < 0$, then the disease is eliminated from the population.

Observations: $R_0 = \frac{\beta N}{\gamma}$

- βN is the rate at which a single infective introduced into a susceptible population of size *N* makes infectious contact. This comes from the term βIS in the differential equations, with I=1 and S=N.
- $\frac{1}{\gamma}$ is the expected length of time that an infective person remains infectious.

Thus, $\frac{\beta N}{\gamma}$ is the expected number of infectious contacts made by an infective.

Thus, if R_o>1, then the disease will remain in the population, and if R_o<1, the disease will die out.

Students are now ready for the West Nile module.