Chapter 5. Integration

§1. The Riemann Integral

Let a and b be two real numbers with $a < b$. Then $[a, b]$ is a closed and bounded interval in $\mathbb{R}$. By a **partition** P of $[a, b]$ we mean a finite ordered set $\{t_0, t_1, \ldots, t_n\}$ such that

$$a = t_0 < t_1 < \cdots < t_n = b.$$

The **norm** of P is defined by $\|P\| := \max\{t_i - t_{i-1} : i = 1, 2, \ldots, n\}$.

Suppose f is a bounded real-valued function on $[a, b]$. Given a partition $\{t_0, t_1, \ldots, t_n\}$ of $[a, b]$, for each $i = 1, 2, \ldots, n$, let

$$m_i := \inf\{f(x) : t_{i-1} \le x \le t_i\} \quad \text{and} \quad M_i := \sup\{f(x) : t_{i-1} \le x \le t_i\}.$$

The **upper sum** $U(f, P)$ and the **lower sum** $L(f, P)$ for the function f and the partition P are defined by

$$U(f, P) := \sum_{i=1}^{n} M_i(t_i - t_{i-1}) \quad \text{and} \quad L(f, P) := \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

The **upper integral** $U(f)$ of f over $[a, b]$ is defined by

$$U(f) := \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$$

and the **lower integral** $L(f)$ of f over $[a, b]$ is defined by

$$L(f) := \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

A bounded function f on $[a, b]$ is said to be (Riemann) **integrable** if $L(f) = U(f)$. In this case, we write

$$\int_a^b f(x)\, dx = L(f) = U(f).$$

By convention we define

$$\int_b^a f(x)\, dx := -\int_a^b f(x)\, dx \quad \text{and} \quad \int_a^a f(x)\, dx := 0.$$

A constant function on $[a, b]$ is integrable. Indeed, if $f(x) = c$ for all $x \in [a, b]$, then $L(f, P) = c(b - a)$ and $U(f, P) = c(b - a)$ for any partition P of $[a, b]$. It follows that

$$\int_a^b c\, dx = c(b - a).$$

Let f be a bounded function from $[a, b]$ to $\mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Suppose that $P = \{t_0, t_1, \ldots, t_n\}$ is a partition of $[a, b]$, and that P_1 is a partition obtained from P by adding one more point $t^* \in (t_{i-1}, t_i)$ for some i. The lower sums for P and P_1 are the same except for the terms involving t_{i-1} or t_i. Let $m_i := \inf\{f(x) : t_{i-1} \leq x \leq t_i\}$, $m' := \inf\{f(x) : t_{i-1} \leq x \leq t^*\}$, and $m'' := \inf\{f(x) : t^* \leq x \leq t_i\}$. Then

$$L(f, P_1) - L(f, P) = m'(t^* - t_{i-1}) + m''(t_i - t^*) - m_i(t_i - t_{i-1}).$$

Since $m' \geq m_i$ and $m'' \geq m_i$, we have $L(f, P) \leq L(f, P_1)$. Moreover, $m' - m \leq 2M$ and $m'' - m \leq 2M$. It follows that

$$m'(t^* - t_{i-1}) + m''(t_i - t^*) - m_i(t_i - t_{i-1}) \leq 2M(t_i - t_{i-1}).$$

Consequently,

$$L(f, P_1) - 2M\|P\| \leq L(f, P) \leq L(f, P_1).$$

Now suppose that P_N is a mesh obtained from P by adding N points. An induction argument shows that

$$L(f, P_N) - 2MN\|P\| \leq L(f, P) \leq L(f, P_N). \tag{1}$$

Similarly we have

$$U(f, P_N) \leq U(f, P) \leq U(f, P_N) + 2MN\|P\|. \tag{2}$$

By the definition of $L(f)$ and $U(f)$, for each $n \in \mathbb{N}$ there exist partitions P and Q of $[a, b]$ such that

$$L(f) - 1/n \leq L(f, P) \quad \text{and} \quad U(f) + 1/n \geq U(f, Q).$$

Consider the partition $P \cup Q$ of $[a, b]$. Since $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$, by (1) and (2) we get

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q).$$

It follows that $L(f) - 1/n \leq U(f) + 1/n$ for all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we obtain $L(f) \leq U(f)$.

We are in a position to establish the following criterion for a bounded function to be integrable.

Theorem 1.1. *A bounded function f on $[a,b]$ is integrable if and only if for each $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that*

$$U(f,P) - L(f,P) < \varepsilon.$$

Proof. Suppose that f is integrable on $[a,b]$. For $\varepsilon > 0$, there exist partitions P_1 and P_2 such that

$$L(f,P_1) > L(f) - \frac{\varepsilon}{2} \quad \text{and} \quad U(f,P_2) < U(f) + \frac{\varepsilon}{2}.$$

For $P := P_1 \cup P_2$ we have

$$L(f) - \frac{\varepsilon}{2} < L(f,P_1) \le L(f,P) \le U(f,P) \le U(f,P_2) < U(f) + \frac{\varepsilon}{2}.$$

Since $L(f) = U(f)$, it follows that $U(f,P) - L(f,P) < \varepsilon$.

Conversely, suppose that for each $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$. Then $U(f,P) < L(f,P) + \varepsilon$. It follows that

$$U(f) \le U(f,P) < L(f,P) + \varepsilon \le L(f) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $U(f) \le L(f)$. But $L(f) \le U(f)$. Therefore $U(f) = L(f)$; that is, f is integrable. $\square$

Let f be a bounded real-valued function on $[a,b]$ and let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a,b]$. For each $i = 1, 2, \ldots, n$, choose $\xi_i \in [x_{i-1}, x_i]$. The sum

$$\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})$$

is called a **Riemann sum** of f with respect to the partition P and points $\{\xi_1, \ldots, \xi_n\}$.

Theorem 1.2. *Let f be a bounded real-valued function on $[a,b]$. Then f is integrable on $[a,b]$ if and only if there exists a real number I with the following property: For any $\varepsilon > 0$ there exists some $\delta > 0$ such that*

$$\left| \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) - I \right| \le \varepsilon \tag{3}$$

whenever $P = \{t_0, t_1, \ldots, t_n\}$ is a partition of $[a,b]$ with $\|P\| < \delta$ and $\xi_i \in [t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$. If this is the case, then

$$\int_a^b f(x)\, dx = I.$$

Proof. Let ε be an arbitrary positive number. Suppose that (3) is true for some partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ and points $\xi_i \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$. Then

$$L(f, P) = \inf\left\{\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) : \xi_i \in [x_{i-1}, x_i] \text{ for } i = 1, 2, \ldots, n\right\} \geq I - \varepsilon$$

and

$$U(f, P) = \sup\left\{\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) : \xi_i \in [x_{i-1}, x_i] \text{ for } i = 1, 2, \ldots, n\right\} \leq I + \varepsilon.$$

It follows that $U(f, P) - L(f, P) \leq 2\varepsilon$. By Theorem 1.1 we conclude that f is integrable on $[a, b]$. Moreover, $L(f) = U(f) = I$.

Conversely, suppose that f is integrable on $[a, b]$. Let $M := \sup\{|f(x)| : x \in [a, b]\}$ and $I := L(f) = U(f)$. Given an arbitrary $\varepsilon > 0$, there exists a partition Q of $[a, b]$ such that $L(f, Q) > I - \varepsilon/2$ and $U(f, Q) < I + \varepsilon/2$. Suppose that Q has N points. Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$ with $\|P\| < \delta$. Consider the partition $P \cup Q$ of $[a, b]$. By (1) and (2) we have

$$L(f, P) \geq L(f, P \cup Q) - 2MN\delta \quad \text{and} \quad U(f, P) \leq U(f, P \cup Q) + 2MN\delta.$$

But $L(f, P \cup Q) \geq L(f, Q) > I - \varepsilon/2$ and $U(f, P \cup Q) \leq U(f, Q) < I + \varepsilon/2$. Choose $\delta := \varepsilon/(4MN)$. Since $\|P\| < \delta$, we deduce from the foregoing inequalities that

$$I - \varepsilon < L(f, P) \leq U(f, P) < I + \varepsilon.$$

Thus, with $\xi_i \in [t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$ we obtain

$$I - \varepsilon < L(f, P) \leq \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \leq U(f, P) < I + \varepsilon.$$

This completes the proof. □

Theorem 1.3. *Let f be a bounded function from a bounded closed interval $[a, b]$ to $\mathbb{R}$. If the set of discontinuities of f is finite, then f is integrable on $[a, b]$.*

Proof. Let D be the set of discontinuities of f. By our assumption, D is finite. So the set $D \cup \{a, b\}$ can be expressed as $\{d_0, d_1, \ldots, d_N\}$ with $a = d_0 < d_1 < \cdots < d_N = b$. Let $M := \sup\{|f(x)| : x \in [a, b]\}$. For an arbitrary positive number ε, we choose $\eta > 0$ such that $\eta < \varepsilon/(8MN)$ and $\eta < (d_j - d_{j-1})/3$ for all $j = 1, \ldots, N$. For $j = 0, 1, \ldots, N$, let $x_j := d_j - \eta$ and $y_j := d_j + \eta$. Then we have

$$a = d_0 < y_0 < x_1 < d_1 < y_1 < \cdots < x_N < d_N = b.$$

Let E be the union of the intervals $[d_0, y_0]$, $[x_1, d_1]$, $[d_1, y_1], \ldots, [x_{N-1}, d_{N-1}]$, $[d_{N-1}, y_{N-1}]$, and $[x_N, d_N]$. There are $2N$ intervals in total. For $j = 1, \ldots, N$, let $F_j := [y_{j-1}, x_j]$. Further, let $F := \cup_{j=1}^N F_j$. The function f is continuous on F, which is a finite union of bounded closed intervals. Hence f is uniformly continuous on F. There exists some $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/(2(b-a))$ whenever $x, y \in F$ satisfying $|x - y| < \delta$. For each $j \in \{1, \ldots, N\}$, let P_j be a partition of F_j such that $\|P_j\| < \delta$. Let

$$P := \{a, b\} \cup D \cup \left(\cup_{j=1}^N P_j\right).$$

The set P can be arranged as $\{t_0, t_1, \ldots, t_n\}$ with $a = t_0 < t_1 < \cdots < t_n = b$. Consider

$$U(f, P) - L(f, P) = \sum_{i=1}^n (M_i - m_i)(t_i - t_{i-1}),$$

where $M_i := \sup\{f(x) : t_{i-1} \le x \le t_i\}$ and $m_i := \inf\{f(x) : t_{i-1} \le x \le t_i\}$. Each interval $[t_{i-1}, t_i]$ is either contained in E or in F, but not in both. Hence

$$\sum_{i=1}^n (M_i - m_i)(t_i - t_{i-1}) = \sum_{[t_{i-1}, t_i] \subseteq E} (M_i - m_i)(t_i - t_{i-1}) + \sum_{[t_{i-1}, t_i] \subseteq F} (M_i - m_i)(t_i - t_{i-1}).$$

There are $2N$ intervals $[t_{i-1}, t_i]$ contained in E. Each interval has length $\eta < \varepsilon/(8MN)$. Noting that $M_i - m_i \le 2M$, we obtain

$$\sum_{[t_{i-1}, t_i] \subseteq E} (M_i - m_i)(t_i - t_{i-1}) \le 2N(2M)\eta < \frac{\varepsilon}{2}.$$

If $[t_{i-1}, t_i] \subseteq F$, then $t_i - t_{i-1} < \delta$; hence $M_i - m_i < \varepsilon/(2(b-a))$. Therefore,

$$\sum_{[t_{i-1}, t_i] \subseteq F} (M_i - m_i)(t_i - t_{i-1}) \le \frac{\varepsilon}{2(b-a)} \sum_{[t_{i-1}, t_i] \subseteq F} (t_i - t_{i-1}) < \frac{\varepsilon}{2(b-a)}(b-a) = \frac{\varepsilon}{2}.$$

From the above estimates we conclude that $U(f, P) - L(f, P) < \varepsilon$. By Theorem 1.1, the function f is integrable on $[a, b]$. □

Example 1. Let $[a, b]$ be a closed interval with $a < b$, and let f be the function on $[a, b]$ given by $f(x) = x$. By Theorem 1.3, f is integrable on $[a, b]$. Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$ and choose $\xi_i := (t_{i-1} + t_i)/2 \in [t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$. Then

$$\sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}) = \frac{1}{2} \sum_{i=1}^n (t_i + t_{i-1})(t_i - t_{i-1}) = \frac{1}{2} \sum_{i=1}^n (t_i^2 - t_{i-1}^2) = \frac{1}{2}(t_n^2 - t_0^2) = \frac{1}{2}(b^2 - a^2).$$

By Theorem 1.2 we have

$$\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2).$$

More generally, for a positive integer k, let f_k be the function given by $f_k(x) = x^k$ for $x \in [a,b]$. Choose

$$\xi_i := \left(\frac{t_{i-1}^k + t_{i-1}^{k-1} t_i + \cdots + t_i^k}{k+1} \right)^{1/k}, \quad i = 1, 2, \ldots, n.$$

We have $t_{i-1} \le \xi_i \le t_i$ for $i = 1, 2, \ldots, n$. Moreover,

$$\sum_{i=1}^n f_k(\xi_i)(t_i - t_{i-1}) = \frac{1}{k+1} \sum_{i=1}^n (t_i^{k+1} - t_{i-1}^{k+1}) = \frac{1}{k+1}(t_n^{k+1} - t_0^{k+1}) = \frac{1}{k+1}(b^{k+1} - a^{k+1}).$$

By Theorem 1.2 we conclude that

$$\int_a^b x^k\,dx = \frac{1}{k+1}(b^{k+1} - a^{k+1}).$$

Example 2. Let g be the function on $[0,1]$ defined by $g(x) := \cos(1/x)$ for $0 < x \le 1$ and $g(0) := 0$. The only discontinuity point of g is 0. By Theorem 1.3, g is integrable on $[0,1]$. Note that g is not uniformly continuous on $(0,1)$. Indeed, let $x_n := 1/(2n\pi)$ and $y_n := 1/(2n\pi + \pi/2)$ for $n \in \mathbb{N}$. Then $\lim_{n\to\infty}(x_n - y_n) = 0$. But

$$|f(x_n) - f(y_n)| = |\cos(2n\pi) - \cos(2n\pi + \pi/2)| = 1 \quad \forall\, n \in \mathbb{N}.$$

Hence g is not uniformly continuous on $(0,1)$. On the other hand, the function u given by $u(x) := 1/x$ for $0 < x \le 1$ and $u(0) := 0$ is not integrable on $[0,1]$, even though u is continuous on $(0,1]$. Theorem 1.3 is not applicable to u, because u is unbounded.

Example 3. Let h be the function on $[0,1]$ defined by $h(x) := 1$ if x is a rational number in $[0,1]$ and $h(x) := 0$ if x is an irrational number in $[0,1]$. Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0,1]$. For $i = 1, \ldots, n$ we have

$$m_i := \inf\{h(x) : x \in [t_{i-1}, t_i]\} = 0 \quad \text{and} \quad M_i := \sup\{h(x) : x \in [t_{i-1}, t_i]\} = 1.$$

Hence $L(h,P) = 0$ and $U(h,P) = 1$ for every partition P of $[0,1]$. Consequently, $L(h) = 0$ and $U(h) = 1$. This shows that h is not Riemann integrable on $[0,1]$.

§2. Properties of the Riemann Integral

In this section we establish some basic properties of the Riemann integral.

Theorem 2.1. *Let f and g be integrable functions from a bounded closed interval $[a, b]$ to $\mathbb{R}$. Then*

(1) *For any real number c, cf is integrable on $[a, b]$ and $\int_a^b (cf)(x)\,dx = c\int_a^b f(x)\,dx$;*

(2) *$f + g$ is integrable on $[a, b]$ and $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.*

Proof. Suppose that f and g are integrable functions on $[a, b]$. Write $I(f) := \int_a^b f(x)\,dx$ and $I(g) := \int_a^b g(x)\,dx$. Let ε be an arbitrary positive number. By Theorem 1.2, there exists some $\delta > 0$ such that

$$\left|\sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}) - I(f)\right| \le \varepsilon \quad \text{and} \quad \left|\sum_{i=1}^n g(\xi_i)(t_i - t_{i-1}) - I(g)\right| \le \varepsilon$$

whenever $P = \{t_0, t_1, \ldots, t_n\}$ is a partition of $[a, b]$ with $\|P\| < \delta$ and $\xi_i \in [t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$. It follows that

$$\left|\sum_{i=1}^n (cf)(\xi_i)(t_i - t_{i-1}) - cI(f)\right| = |c|\left|\sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}) - I(f)\right| \le |c|\varepsilon.$$

Hence cf is integrable on $[a, b]$ and $\int_a^b (cf)(x)\,dx = c\int_a^b f(x)\,dx$. Moreover,

$$\begin{aligned}&\left|\sum_{i=1}^n (f+g)(\xi_i)(t_i - t_{i-1}) - [I(f) + I(g)]\right| \\ &\le \left|\sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}) - I(f)\right| + \left|\sum_{i=1}^n g(\xi_i)(t_i - t_{i-1}) - I(g)\right| \le 2\varepsilon.\end{aligned}$$

Therefore $f + g$ is integrable on $[a, b]$ and $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. □

Theorem 2.2. *Let f and g be integrable functions on $[a, b]$. Then fg is an integrable function on $[a, b]$.*

Proof. Let us first show that f^2 is integrable on $[a, b]$. Since f is bounded, there exists some $M > 0$ such that $|f(x)| \le M$ for all $x \in [a, b]$. It follows that

$$\big|[f(x)]^2 - [f(y)]^2\big| = |f(x) + f(y)||f(x) - f(y)| \le 2M|f(x) - f(y)| \quad \text{for all } x, y \in [a, b].$$

We deduce from the above inequality that $U(f^2, P) - L(f^2, P) \le 2M[U(f, P) - L(f, P)]$ for any partition P of $[a, b]$. Let $\varepsilon > 0$. Since f is integrable on $[a, b]$, by Theorem 1.1

there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon/(2M)$. Consequently, $U(f^2,P) - L(f^2,P) < \varepsilon$. By Theorem 1.1 again we conclude that f^2 is integrable on $[a,b]$.

Note that $fg = [(f+g)^2 - (f-g)^2]/4$. By Theorem 2.1, $f+g$ and $f-g$ are integrable on $[a,b]$. By what has been proved, both $(f+g)^2$ and $(f-g)^2$ are integrable on $[a,b]$. Using Theorem 2.1 again, we conclude that fg is integrable on $[a,b]$. □

Theorem 2.3. *Let a,b,c,d be real numbers such that $a \le c < d \le b$. If a real-valued function f is integrable on $[a,b]$, then $f|_{[c,d]}$ is integrable on $[c,d]$.*

Proof. Suppose that f is integrable on $[a,b]$. Let ε be an arbitrary positive number. By Theorem 1.1, there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$. It follows that $U(f, P \cup \{c,d\}) - L(f, P \cup \{c,d\}) < \varepsilon$. Let $Q := (P \cup \{c,d\}) \cap [c,d]$. Then Q is a partition of $[c,d]$. We have

$$U(f|_{[c,d]}, Q) - L(f|_{[c,d]}, Q) \le U(f, P \cup \{c,d\}) - L(f, P \cup \{c,d\}) < \varepsilon.$$

Hence $f|_{[c,d]}$ is integrable on $[c,d]$. □

Theorem 2.4. *Let f be a bounded real-valued function on $[a,b]$. If $a < c < b$, and if f is integrable on $[a,c]$ and $[c,b]$, then f is integrable on $[a,b]$ and*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof. Suppose that f is integrable on $[a,c]$ and $[c,b]$. We write $I_1 := \int_a^c f(x)\,dx$ and $I_2 := \int_c^b f(x)\,dx$. Let $\varepsilon > 0$. By Theorem 1.1, there exist a partition $P_1 = \{s_0, s_1, \ldots, s_m\}$ of $[a,c]$ and a partition $P_2 = \{t_0, t_1, \ldots, t_n\}$ of $[c,b]$ such that

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2} \quad \text{and} \quad U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{2}.$$

Let $P := P_1 \cup P_2 = \{s_0, \ldots, s_{m-1}, t_0, \ldots, t_n\}$. Then P is a partition of $[a,b]$. We have

$$L(f) \ge L(f,P) = L(f,P_1) + L(f,P_2) > U(f,P_1) + U(f,P_2) - \varepsilon \ge I_1 + I_2 - \varepsilon$$

and

$$U(f) \le U(f,P) = U(f,P_1) + U(f,P_2) < L(f,P_1) + L(f,P_2) + \varepsilon \le I_1 + I_2 + \varepsilon$$

It follows that

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \varepsilon < L(f) \le U(f) < \int_a^c f(x)\,dx + \int_c^b f(x)\,dx + \varepsilon.$$

Since the above inequalities are valid for all $\varepsilon > 0$, we conclude that f is integrable on $[a, b]$ and $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. □

Let a, b, c be real numbers in any order, and let J be a bounded closed interval containing a, b, and c. If f is integrable on J, then by Theorems 2.3 and 2.4 we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Theorem 2.5. *Let f and g be integrable functions on $[a, b]$. If $f(x) \le g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.*

Proof. By Theorem 2.1, $h := g - f$ is integrable on $[a, b]$. Since $h(x) \ge 0$ for all $x \in [a, b]$, it is clear that $L(h, P) \ge 0$ for any partition P of $[a, b]$. Hence, $\int_a^b h(x)\,dx = L(h) \ge 0$. Applying Theorem 2.1 again, we see that

$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx = \int_a^b h(x)\,dx \ge 0. \qquad \square$$

Theorem 2.6. *If f is an integrable function on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and*

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Proof. Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. For each $i \in \{1, \ldots, n\}$, let M_i and m_i denote the supremum and infimun respectively of f on $[t_{i-1}, t_i]$, and let M_i^* and m_i^* denote the supremum and infimun respectively of $|f|$ on $[t_{i-1}, t_i]$. Then

$$M_i - m_i = \sup\{f(x) - f(y) : x, y \in [t_{i-1}, t_i]\}$$

and

$$M_i^* - m_i^* = \sup\{|f(x)| - |f(y)| : x, y \in [t_{i-1}, t_i]\}.$$

By the triangle inequality, $\big||f(x)| - |f(y)|\big| \le |f(x) - f(y)|$. Hence $M_i^* - m_i^* \le M_i - m_i$ and

$$\sum_{i=1}^{n} (M_i^* - m_i^*)(t_i - t_{i-1}) \le \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}).$$

It follows that $U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P)$. Let ε be an arbitrary positive number. By our assumption, f is integrable on $[a, b]$. By Theorem 1.1, there exists a partition P such that $U(f, P) - L(f, P) < \varepsilon$. Hence $U(|f|, P) - L(|f|, P) < \varepsilon$. By using Theorem 1.1 again we conclude that $|f|$ is integrable on $[a, b]$. Furthermore, since $f(x) \le |f(x)|$ and $-f(x) \le |f(x)|$ for all $x \in [a, b]$, by Theorem 2.5 we have

$$\int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \quad \text{and} \quad -\int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx.$$

Therefore $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$. □

§3. Fundamental Theorem of Calculus

In this section we give two versions of the Fundamental Theorem of Calculus and their applications.

Let f be a real-valued function on an interval I. A function F on I is called an **antiderivative** of f on I if $F'(x) = f(x)$ for all $x \in I$. If F is an antiderivative of f, then so is $F + C$ for any constant C. Conversely, if F and G are antiderivatives of f on I, then $G'(x) - F'(x) = 0$ for all $x \in I$. Thus, there exists a constant C such that $G(x) - F(x) = C$ for all $x \in I$. Consequently, $G = F + C$.

The following is the first version of the Fundamental Theorem of Calculus.

Theorem 3.1. *Let f be an integrable function on $[a, b]$. If F is a continuous function on $[a, b]$ and if F is an antiderivative of f on (a, b), then*

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b := F(b) - F(a).$$

Proof. Let $\varepsilon > 0$. By Theorem 1.1, there exists a partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. Since $t_0 = a$ and $t_n = b$ we have

$$F(b) - F(a) = \sum_{i=1}^{n} [F(t_i) - F(t_{i-1})].$$

By the Mean Value Theorem, for each $i \in \{1, \ldots, n\}$ there exists $x_i \in (t_{i-1}, t_i)$ such that

$$F(t_i) - F(t_{i-1}) = F'(x_i)(t_i - t_{i-1}) = f(x_i)(t_i - t_{i-1}).$$

Consequently,

$$L(f, P) \le F(b) - F(a) = \sum_{i=1}^{n} f(x_i)(t_i - t_{i-1}) \le U(f, P).$$

On the other hand,

$$L(f, P) \le \int_a^b f(x)\,dx \le U(f, P).$$

Thus both $F(b) - F(a)$ and $\int_a^b f(x)\,dx$ lie in $[L(f, P), U(f, P)]$ with $U(f, P) - L(f, P) < \varepsilon$. Hence

$$\left| [F(b) - F(a)] - \int_a^b f(x)\,dx \right| < \varepsilon.$$

Since the above inequality is valid for all $\varepsilon > 0$, we obtain $\int_a^b f(x)\,dx = F(b) - F(a)$. □

Example 1. Let k be a positive integer. Find $\int_a^b x^k\,dx$.

Solution. We know that the function $g_k : x \mapsto x^{k+1}/(k+1)$ is an antiderivative of the function $f_k : x \mapsto x^k$. By the Fundamental Theorem of Calculus we obtain

$$\int_a^b x^k\,dx = \frac{x^{k+1}}{k+1}\bigg|_a^b = \frac{b^{k+1} - a^{k+1}}{k+1}.$$

Example 2. Find the integral $\int_1^2 1/x\,dx$.

Solution. On the interval $(0, \infty)$, the function $x \mapsto \ln x$ is an antiderivative the function $x \mapsto 1/x$. By the Fundamental Theorem of Calculus we obtain

$$\int_1^2 \frac{1}{x}\,dx = \ln x\Big|_1^2 = \ln 2 - \ln 1 = \ln 2.$$

This integral can be used to find the limit

$$\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right).$$

Indeed, let $f(x) := 1/x$ for $x = [1,2]$, and let $t_i = 1 + i/n$ for $i = 0, 1, \ldots, n$. Then $P := \{t_0, t_1, \ldots, t_n\}$ is a partition of $[1,2]$ and

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$$

is a Riemann sum of f with respect to P and points $\{t_1, \ldots, t_n\}$. By Theorem 1.2 we get

$$\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)(t_i - t_{i-1}) = \int_1^2 \frac{1}{x}\,dx = \ln 2.$$

Example 3. A curve in plane is represented by a continuous mapping $u = (u_1, u_2)$ from $[a,b]$ to $\mathbb{R}^2$. We use $L(u)$ to denote the length of the curve. Suppose that u_1' and u_2' are continuous on $[a,b]$. Then u is rectifiable. For $t \in [a,b]$, let $s(t)$ denote the length of the curve $u|_{[a,t]}$. It was proved in Theorem 7.1 of Chapter 4 that

$$s'(t) = \sqrt{[u_1'(t)]^2 + [u_2'(t)]^2}, \quad t \in [a,b].$$

By Theorem 3.1 (the Fundamental Theorem of Calculus), we obtain

$$L(u) = s(b) - s(a) = \int_a^b s'(t)\,dt = \int_a^b \sqrt{[u_1'(t)]^2 + [u_2'(t)]^2}\,dt.$$

The following is the second version of the Fundamental Theorem of Calculus.

Theorem 3.2. *Let f be an integrable function on $[a, b]$. Define*

$$F(x) := \int_a^x f(t)\,dt, \quad x \in [a, b].$$

Then F is a continuous function on $[a, b]$. Furthermore, if f is continuous at a point $c \in [a, b]$, then F is differentiable at c and

$$F'(c) = f(c).$$

Proof. Since f is bounded on $[a, b]$, there exists a constant $M > 0$ such that $|f(x)| \le M$ for all $x \in [a, b]$. If $x, y \in [a, b]$ and $x < y$, then

$$F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt = \int_x^y f(t)\,dt.$$

Since $-M \le f(t) \le M$ for $x \le t \le y$, by Theorem 2.5 we have

$$-M(y - x) \le \int_x^y f(t)\,dt \le M(y - x).$$

It follows that $|F(y) - F(x)| \le M|y - x|$. For given $\varepsilon > 0$, choose $\delta = \varepsilon/M$. Then $|y - x| < \delta$ implies $|F(y) - F(x)| \le M|y - x| < \varepsilon$. This shows that F is continuous on $[a, b]$.

Now suppose that f is continuous at $c \in [a, b)$. Let $h > 0$. By Theorem 2.4 we have

$$\frac{F(c + h) - F(c)}{h} - f(c) = \frac{1}{h} \int_c^{c+h} f(t)\,dt - f(c) = \frac{1}{h} \int_c^{c+h} \big[f(t) - f(c)\big]\,dt.$$

Let $\varepsilon > 0$ be given. Since f is continuous at c, there exists some $\delta > 0$ such that $|f(t) - f(c)| \le \varepsilon$ whenever $c \le t \le c + \delta$. Therefore, if $0 < h < \delta$, then

$$\left|\frac{F(c + h) - F(c)}{h} - f(c)\right| = \left|\frac{1}{h} \int_c^{c+h} \big[f(t) - f(c)\big]\,dt\right| \le \frac{1}{h} \int_c^{c+h} \big|f(t) - f(c)\big|\,dt \le \varepsilon.$$

Consequently,

$$\lim_{h \to 0^+} \frac{F(c + h) - F(c)}{h} = f(c).$$

Similarly, if f is continuous at $c \in (a, b]$, then

$$\lim_{h \to 0^-} \frac{F(c + h) - F(c)}{h} = f(c).$$

This completes the proof of the theorem. □

Example 4. Let f be a continuous function on $[a, b]$, and let $F(x) := \int_x^b f(t)\,dt$ for each $x \in [a, b]$. Then we have

$$F(x) = \int_x^b f(t)\,dt = -\int_b^x f(t)\,dt.$$

By Theorem 3.2, F is differentiable on $[a, b]$ and $F'(x) = -f(x)$ for $a \le x \le b$.

Example 5. Let $F(x) := \int_{-x}^{x^2} \sqrt{4 + t^2}\,dt$, $x \in \mathbb{R}$. Find $F'(x)$ for $x \in \mathbb{R}$.

Solution. We have

$$F(x) = \int_{-x}^{0} \sqrt{4 + t^2}\,dt + \int_0^{x^2} \sqrt{4 + t^2}\,dt = -\int_0^{-x} \sqrt{4 + t^2}\,dt + \int_0^{x^2} \sqrt{4 + t^2}\,dt.$$

By using the chain rule and Theorem 3.2 we obtain

$$F'(x) = \sqrt{4 + x^2} + 2x\sqrt{4 + x^4}.$$

Example 6. Let $G(x) := \int_2^x x\cos(t^3)\,dt$, $x \in \mathbb{R}$. Find $G''(x)$ for $x \in \mathbb{R}$.

Solution. We have $G(x) = x\int_2^x \cos(t^3)\,dt$. By Theorem 3.2 and the product rule for differentiation, we obtain

$$G'(x) = \int_2^x \cos(t^3)\,dt + x\cos(x^3).$$

Taking derivative once more, we get

$$G''(x) = \cos(x^3) + \cos(x^3) + x[-\sin(x^3)](3x^2) = 2\cos(x^3) - 3x^3\sin(x^3).$$

§4. Indefinite Integrals

An antiderivative of a function f is also called an **indefinite integral** of f. It is customary to denote an indefinite integral of f by

$$\int f(x)\,dx.$$

For example, for $\mu \in \mathbb{R} \setminus \{-1\}$ we have

$$\int x^\mu\,dx = \frac{x^{\mu+1}}{\mu + 1} + C, \quad x \in (0, \infty).$$

If $\mu \in \mathbb{N}_0$, then the above formula is valid for all $x \in \mathbb{R}$. If $\mu \in \mathbb{Z}$ and $\mu \leq -2$, then the formula holds for $x \in (-\infty, 0) \cup (0, \infty)$. For $\mu = -1$ we have

$$\int \frac{1}{x}\, dx = \ln|x| + C, \quad x \in (-\infty, 0) \cup (0, \infty).$$

The following formulas for integration are easily derived from the corresponding formulas for differentiation:

$$\int e^x\, dx = e^x + C, \quad x \in (-\infty, \infty).$$

$$\int \cos x\, dx = \sin x + C, \quad x \in (-\infty, \infty),$$

$$\int \sin x\, dx = -\cos x + C, \quad x \in (-\infty, \infty),$$

$$\int \frac{1}{1+x^2}\, dx = \arctan x + C \quad x \in (-\infty, \infty),$$

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + C \quad x \in (-1, 1).$$

If F_1 and F_2 are differentiable functions on an interval, and if $F_1' = f_1$ and $F_2' = f_2$, then for $c_1, c_2 \in \mathbb{R}$ we have

$$[c_1 F_1 + c_2 F_2]' = c_1 F_1' + c_2 F_2' = c_1 f_1 + c_2 f_2.$$

It follows that

$$\int [c_1 f_1(x) + c_2 f_2(x)]\, dx = c_1 \int f_1(x)\, dx + c_2 \int f_2(x)\, dx.$$

Now let u and v be differentiable functions on an interval. By the product rule for differentiation we have

$$(uv)' = u'v + uv'.$$

From this we deduce the following formula for integration by parts:

$$\int u(x)v'(x)\, dx = u(x)v(x) - \int u'(x)v(x)\, dx.$$

It can also be written as

$$\int u\, dv = uv - \int v\, du.$$

Example 1. Find $\int x^2 e^x \, dx$.

Solution. By integration by parts we have

$$\int x^2 e^x \, dx = \int x^2 d(e^x) = x^2 e^x - \int e^x d(x^2) = x^2 e^x - 2\int x e^x \, dx.$$

By using integration by parts again we obtain

$$\int x e^x \, dx = \int x d(e^x) = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Therefore

$$\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C.$$

In general, if p is a polynomial, then

$$\int p(x) e^x \, dx = \int p(x) d(e^x) = p(x) e^x - \int p'(x) e^x \, dx,$$

where the degree of p' is one less than that of p. Thus the integral $\int p(x) e^x$ can be computed by using integration by parts repeatedly. This method also applies to the integrals $\int p(x) \sin x \, dx$ and $\int p(x) \cos x \, dx$.

Example 2. Find $\int x \ln x \, dx$.

Solution. Integration by parts gives

$$\begin{aligned} \int x \ln x \, dx &= \int \ln x \, d(x^2/2) = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \, d(\ln x) \\ &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} \, dx = \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + C. \end{aligned}$$

In general, if p is a polynomial given by $p(x) = \sum_{k=0} a_k x^k$, then

$$\int p(x) \, dx = \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1} + C.$$

Let $s(x) := \sum_{k=0}^{n} a_k x^{k+1}/(k+1)$. By using integration by parts we get

$$\int p(x) \ln x \, dx = \int \ln x d(s(x)) = s(x) \ln x - \int s(x) \, d(\ln x) = s(x) \ln x - \int \frac{s(x)}{x} \, dx.$$

This method also applies to the integral $\int p(x) \arctan x \, dx$.

Let u be a differentiable function from an interval I to an interval J, and let F be a differentiable function from J to $\mathbb{R}$. Suppose $F' = f$. By the chain rule the composition $F \circ u$ is differentiable on I and

$$(F \circ u)'(x) = F'(u(x))u'(x) = f(u(x))u'(x), \quad x \in I.$$

Thus we have the following formula for change of variables in an integral:

$$\int f(u(x))u'(x)\,dx = F(u(x)) + C.$$

Example 3. Find $\int \sin^2 x \cos x\,dx$.

Solution. Let $u := \sin x$. Then $du = \cos x\,dx$. Hence

$$\int \sin^2 x \cos x\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C.$$

We can use this integral together with the identity $\sin^2 x + \cos^2 x = 1$ to find the integral $\int \cos^3 x\,dx$:

$$\begin{aligned}\int \cos^3 x\,dx &= \int \cos^2 x \cos x\,dx = \int (1 - \sin^2 x)\cos x\,dx \\ &= \int \cos x\,dx - \int \sin^2 x \cos x\,dx = \sin x - \frac{1}{3}\sin^3 x + C.\end{aligned}$$

For integrals involving sine and cosine, the following double angle formulas will be useful:

$$\sin(2x) = 2\sin x \cos x,$$

$$\cos(2x) = \cos^2 x - \sin^2 x.$$

The second formula together with the identity $\sin^2 x + \cos^2 x = 1$ gives

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos(2x)}{2}.$$

Thus we have

$$\int \sin^2 x\,dx = \int \frac{1}{2}\,dx - \frac{1}{2}\int \cos(2x)\,dx = \frac{x}{2} - \frac{1}{4}\sin(2x) + C.$$

In general, for nonnegative integers m and n, the integral

$$\int \sin^m x \cos^n x\,dx$$

can be calculated as follows: (1) If m is odd, use the substitution $u = \cos x$ and the identity $\sin^2 x = 1 - \cos^2 x$. (2) If n is odd, use the substitution $u = \sin x$ and the identity $\cos^2 x = 1 - \sin^2 x$. (3) If both m and n are even, use $\sin^2 x = (1 - \cos(2x))/2$ and $\cos^2 x = (1 + \cos(2x))/2$ to reduce the exponents of sine and cosine.

Example 4. Find the following integrals:

$$\int \tan x\, dx, \quad \int \cot x\, dx, \quad \int \sec x\, dx, \quad \int \csc x\, dx.$$

Solution. For the first integral we use the substitution $u = \cos x$ and get

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{1}{u}\, du = -\ln|u| + C = -\ln|\cos x| + C = \ln|\sec x| + C.$$

Similarly,

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx = \int \frac{d(\sin x)}{\sin x} = \ln|\sin x| + C.$$

In order to find $\int \sec x\, dx$, we observe that

$$\frac{d}{dx}(\sec x + \tan x) = \sec x \tan x + \sec^2 x = \sec x(\tan x + \sec x).$$

It follows that

$$\int \sec x\, dx = \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} = \ln|\sec x + \tan x| + C.$$

Similarly,

$$\int \csc x\, dx = -\int \frac{d(\csc x + \cot x)}{\csc x + \cot x} = -\ln|\csc x + \cot x| + C.$$

Example 5. For $a > 0$, calculate the following integrals:

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx \quad \text{and} \quad \int \frac{1}{\sqrt{x^2 - a^2}}\, dx.$$

Solution. For the first integral we let $x = a \tan t$ for $-\pi/2 < t < \pi/2$. Then $\sec t > 0$ and $x^2 + a^2 = a^2(\tan^2 t + 1) = a^2 \sec^2 t$. Hence

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \int \frac{a \sec^2 t}{a \sec t}\, dt = \int \sec t\, dt = \ln(\tan t + \sec t).$$

But $\sec t = \sqrt{\tan^2 t + 1}$. Consequently,

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \ln\left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} + 1}\right) + C = \ln\big(x + \sqrt{x^2 + a^2}\big) + C_1,$$

where $C_1 = C - \ln a$. Similarly,

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left|x + \sqrt{x^2 - a^2}\right| + C, \quad |x| > a.$$

Let us consider $\int \sqrt{\alpha x^2 + \beta}\,dx$, where $\alpha, \beta \in \mathbb{R}$. Integrating by parts, we obtain

$$\int \sqrt{\alpha x^2 + \beta}\,dx = x\sqrt{\alpha x^2 + \beta} - \int \frac{\alpha x^2}{\sqrt{\alpha x^2 + \beta}}\,dx.$$

Note that

$$\frac{\alpha x^2}{\sqrt{\alpha x^2 + \beta}} = \frac{\alpha x^2 + \beta - \beta}{\sqrt{\alpha x^2 + \beta}} = \sqrt{\alpha x^2 + \beta} - \frac{\beta}{\sqrt{\alpha x^2 + \beta}}.$$

Hence

$$\int \sqrt{\alpha x^2 + \beta}\,dx = x\sqrt{\alpha x^2 + \beta} - \int \sqrt{\alpha x^2 + \beta}\,dx + \int \frac{\beta}{\sqrt{\alpha x^2 + \beta}}\,dx.$$

It follows that

$$\int \sqrt{\alpha x^2 + \beta}\,dx = \frac{1}{2}x\sqrt{\alpha x^2 + \beta} + \frac{\beta}{2}\int \frac{1}{\sqrt{\alpha x^2 + \beta}}\,dx.$$

In particular, we get

$$\int \sqrt{x^2 + a^2}\,dx = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

and

$$\int \sqrt{x^2 - a^2}\,dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + C$$

For $a > 0$, a simple substitution gives

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C, \quad -a < x < a.$$

Therefore,

$$\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C, \quad -a < x < a.$$

A rational function has the form $p(x)/s(x)$, where p and s are polynomials. There exist unique polynomials q and r such that $p(x) = q(x)s(x) + r(x)$, where the degree of r is less than the degree of s. It follows that

$$\frac{p(x)}{s(x)} = q(x) + \frac{r(x)}{s(x)}.$$

In order to find $\int r(x)/s(x)\,dx$, we decompose $r(x)/s(x)$ as the sum of terms of the following type:

$$\frac{c_1}{x-\alpha} + \cdots + \frac{c_n}{(x-\alpha)^n} + \frac{d_1 + e_1 x}{(x-\beta)^2+\gamma^2} + \cdots + \frac{d_m + e_m x}{[(x-\beta)^2+\gamma^2]^m}.$$

Example 6. For $b, c, \lambda, \mu \in \mathbb{R}$, find the integral

$$\int \frac{\lambda x + \mu}{x^2 + bx + c}\,dx.$$

Solution. We may write

$$\int \frac{\lambda x + \mu}{x^2 + bx + c}\,dx = \int \frac{\lambda}{2}\frac{2x+b}{x^2+bx+c}\,dx + \int \frac{\mu - b\lambda/2}{x^2+bx+c}\,dx.$$

Clearly,

$$\int \frac{\lambda}{2}\frac{2x+b}{x^2+bx+c}\,dx = \frac{\lambda}{2}\ln|x^2+bx+c| + C.$$

So it remains to find the integral $\int dx/(x^2+bx+c)$. There are three possible cases: $b^2-4c>0$, $b^2-4c=0$, and $b^2-4c<0$. If $b^2-4c>0$, then $x^2+bx+c=(x-\alpha)(x-\beta)$, where α and β are distinct real numbers. In this case,

$$\int \frac{1}{(x-\alpha)(x-\beta)}\,dx = \int \frac{1}{\alpha-\beta}\left(\frac{1}{x-\alpha} - \frac{1}{x-\beta}\right)dx = \frac{1}{\alpha-\beta}\big[\ln|x-\alpha| - \ln|x-\beta|\big] + C.$$

If $b^2-4c=0$, then $x^2+bx+c=(x-\alpha)^2$, where $\alpha=-b/2$. In this case,

$$\int \frac{1}{(x-\alpha)^2}\,dx = -\frac{1}{x-\alpha} + C.$$

Finally, if $b^2-4c<0$, we have $x^2+bx+c=(x+b/2)^2+\gamma^2$, where $\gamma=\sqrt{c-b^2/4}$. Thus

$$\int \frac{1}{x^2+bx+c}\,dx = \int \frac{1}{(x+b/2)^2+\gamma^2} = \frac{1}{\gamma}\arctan\frac{x+b/2}{\gamma} + C.$$

§5. Definite Integrals

As an application of the Fundamental Theorem of Calculus, we establish the following formula of integration by parts.

Theorem 5.1. *If u and v are continuous functions on $[a,b]$ that are differentiable on (a,b), and if u' and v' are integrable on $[a,b]$, then*

$$\int_a^b u(x)v'(x)dx + \int_a^b u'(x)v(x)\,dx = u(b)v(b) - u(a)v(a).$$

Proof. Let $F := uv$. Then $F'(x) = u'(x)v(x) + u(x)v'(x)$ for $x \in (a,b)$. By Theorem 3.1 we have

$$\int_a^b F'(x)\,dx = F(b) - F(a) = u(b)v(b) - u(a)v(a). \qquad \square$$

Example 1. Find $\int_0^1 x \ln x\,dx$.

Solution. For $k = 1, 2, \ldots$, let $f_k(x) := x^k \ln x$, $x > 0$. Then f_k is continuous on $(0, \infty)$. Moreover,

$$\lim_{x\to 0^+} x^k \ln x = \lim_{x\to 0^+} \frac{\ln x}{(1/x)^k} = \lim_{y\to +\infty} \frac{\ln(1/y)}{y^k} = \lim_{y\to +\infty} \frac{-\ln y}{y^k} = 0.$$

Thus, by defining $f_k(0) := 0$, f_k is extended to a continuous function on $[0, \infty)$. Integration by parts gives

$$\int_0^1 x \ln x\,dx = \frac{x^2}{2} \ln x \Big|_0^1 - \int_0^1 \frac{x}{2}\,dx = -\frac{1}{4}x^2\Big|_0^1 = -\frac{1}{4}.$$

Now let us consider the integral $\int_0^1 \ln x\,dx$. The function $f_0 : x \mapsto \ln x$ is unbounded on $(0,1)$. So this is an improper integral. We define

$$\int_0^1 \ln x\,dx := \lim_{a\to 0^+} \int_a^1 \ln x\,dx.$$

Integration by parts gives

$$\int_a^1 \ln x\,dx = x \ln x\Big|_a^1 - \int_a^1 dx = -a\ln a - (1-a).$$

Consequently,

$$\int_0^1 \ln x\,dx = \lim_{a\to 0^+} [-a\ln a - (1-a)] = -1.$$

Example 2. For $n = 0, 1, 2, \ldots$, let

$$I_n := \int_0^1 (1-x^2)^n\,dx.$$

Find I_n.

Solution. We have $I_0 = 1$. For $n \ge 1$, integrating by parts, we get

$$I_n = \int_0^1 (1-x^2)^n\,dx = x(1-x^2)^n\Big|_0^1 - \int_0^1 x d\big((1-x^2)^n\big) = 2n\int_0^1 x^2(1-x^2)^{n-1}\,dx.$$

We may write $x^2(1-x^2)^{n-1} = [1-(1-x^2)](1-x^2)^{n-1} = (1-x^2)^{n-1} - (1-x^2)^n$. Hence

$$I_n = 2n\int_0^1 (1-x^2)^{n-1}\,dx - 2n\int_0^1 (1-x^2)^n\,dx = 2nI_{n-1} - 2nI_n.$$

It follows that $(2n+1)I_n = 2nI_{n-1}$. Thus $I_1 = 2/3$. In general,

$$I_n = \frac{2n}{2n+1}I_{n-1} = \frac{2n}{2n+1}\frac{2n-2}{2n-1}\cdots\frac{2}{3} = \prod_{k=1}^{n}\frac{2k}{2k+1}.$$

As another application of the Fundamental Theorem of Calculus, we give the following formula for change of variables in a definite integral.

Theorem 5.2. *Let u be a differentiable function on $[a,b]$ such that u' is integrable on $[a,b]$. If f is continuous on $I := u([a,b])$, then*

$$\int_a^b f(u(t))u'(t)\,dt = \int_{u(a)}^{u(b)} f(x)\,dx.$$

Proof. Since u is continuous, $I = u([a,b])$ is a closed and bounded interval. Also, since $f\circ u$ is continuous and u' is integrable on $[a,b]$, the function $(f\circ u)u'$ is integrable on $[a,b]$. If $I = u([a,b])$ is a single point, then u is constant on $[a,b]$. In this case $u'(t) = 0$ for all $t\in[a,b]$ and both integrals above are zero. Otherwise, for $x\in I$ define

$$F(x) := \int_{u(a)}^{x} f(s)\,ds.$$

Since f is continuous on I, $F'(x) = f(x)$ for all $x\in I$, by Theorem 3.2. By the chain rule we have

$$(F\circ u)'(t) = F'(u(t))u'(t) = f(u(t))u'(t), \quad t\in[a,b].$$

Therefore by Theorem 3.1 we obtain

$$\int_a^b f(u(t))u'(t)\,dt = (F\circ u)(b) - (F\circ u)(a) = F(u(b)) - F(u(a)) = \int_{u(a)}^{u(b)} f(x)\,dx. \qquad \square$$

Example 3. For $a > 0$, find $\int_0^a \sqrt{a^2 - x^2}\, dx$.

Solution. Let $x = a \sin t$. When $t = 0$, $x = 0$. When $t = \pi/2$, $x = a$. By Theorem 5.2 we get

$$\int_0^a \sqrt{a^2 - x^2}\, dx = \int_0^{\pi/2} \sqrt{a^2(1 - \sin^2 t)}\, a \cos t\, dt = \int_0^a a^2 \sqrt{\cos^2 t} \cos t\, dt.$$

Since $\cos t \geq 0$ for $0 \leq t \leq \pi/2$, we have $\sqrt{\cos^2 t} = \cos t$. Thus

$$\int_0^a \sqrt{a^2 - x^2}\, dx = a^2 \int_0^{\pi/2} \cos^2 t\, dt = a^2 \int_0^{\pi/2} \frac{1 + \cos(2t)}{2}\, dt$$
$$= \frac{a^2}{2}\Big(t + \frac{1}{2}\sin(2t)\Big)\Big|_0^{\pi/2} = \frac{\pi}{4}a^2.$$

Example 4. Let $a > 0$. Suppose that f is a continuous function on $[-a, a]$. Prove the following statements.

(1) If f is an even function, *i.e.*, $f(-x) = f(x)$ for all $x \in [0, a]$, then

$$\int_{-a}^a f(x)\, dx = 2 \int_0^a f(x)\, dx.$$

(2) If f is an odd function, *i.e.*, $f(-x) = -f(x)$ for all $x \in [0, a]$, then $\int_{-a}^a f(x)\, dx = 0$.

Proof. We have

$$\int_{-a}^a f(x)\, dx = \int_{-a}^0 f(x)\, dx + \int_0^a f(x)\, dx.$$

In the integral $\int_{-a}^0 f(x)\, dx$ we make the change of variables: $x = -t$. When $t = a$, $x = -a$; when $t = 0$, $x = 0$. By Theorem 5.2 we get

$$\int_{-a}^0 f(x)\, dx = \int_a^0 f(-t) d(-t) = -\int_a^0 f(-t)\, dt = \int_0^a f(-t)\, dt.$$

It follows that

$$\int_{-a}^a f(x)\, dx = \int_0^a f(-t)\, dt + \int_0^a f(t)\, dt = \int_0^a [f(-t) + f(t)]\, dt.$$

If f is an even function, then $f(-t) = f(t)$ for all $t \in [0, a]$; hence

$$\int_{-a}^a f(x)\, dx = 2 \int_0^a f(t)\, dt = 2 \int_0^a f(x)\, dx.$$

If f is an odd function, then $f(-t) = -f(t)$ for all $t \in [0, a]$; hence

$$\int_{-a}^a f(x)\, dx = 0.$$