Chapter 3. Continuous Functions

§1. Limits of Functions

Let E be a subset of $\mathbb{R}$ and c a point of $\mathbb{R}$. We say that c is a **limit point** of E if there exists a sequence $(x_n)_{n=1,2,\ldots}$ in E such that $x_n \neq c$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = c$. The set of all limit points of E is denoted by E'.

For example, if E is the interval $(0, 1]$, then $E' = [0, 1]$. If F is the set $\{1/n : n \in \mathbb{N}\}$, then $F' = \{0\}$.

Let f be a function from a subset E of $\mathbb{R}$ to $\mathbb{R}$ and let c be a limit point of E. We say that a real number L is a **limit of f at** c, and we write $\lim_{x\to c} f(x) = L$, if for each $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$x \in E \text{ and } 0 < |x - c| < \delta \quad \text{imply} \quad |f(x) - L| < \varepsilon.$$

For example, let f be the function from $\mathbb{R}$ to $\mathbb{R}$ given by $f(x) = x$, $x \in \mathbb{R}$ For each $c \in \mathbb{R}$ we have $\lim_{x\to c} f(x) = c$. Indeed, for given $\varepsilon > 0$, choose $\delta = \varepsilon > 0$. Then

$$x \in \mathbb{R} \text{ and } 0 < |x - c| < \delta \quad \text{imply} \quad |f(x) - c| = |x - c| < \varepsilon.$$

Similarly, if $b \in \mathbb{R}$ and g is the function from $\mathbb{R}$ to $\mathbb{R}$ given by $g(x) = b$ for all $x \in \mathbb{R}$, then $\lim_{x\to c} g(x) = b$ for each $c \in \mathbb{R}$.

The following theorem establishes the relationship between limits of functions and limits of sequences.

Theorem 1.1. *Let f be a function from a subset E of $\mathbb{R}$ to $\mathbb{R}$ and let c be a limit point of E. Then $\lim_{x\to c} f(x) = L$ if and only if for every sequence $(x_n)_{n=1,2\ldots}$ in E that converges to c with $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))_{n=1,2,\ldots}$ converges to L.*

Proof. Suppose that $\lim_{x\to c} f(x) = L$. Let $(x_n)_{n=1,2,\ldots}$ be a sequence in E such that $x_n \neq c$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = c$. We wish to show that $\lim_{n\to\infty} f(x_n) = L$. For given $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$x \in E \text{ and } 0 < |x - c| < \delta \quad \text{imply} \quad |f(x) - L| < \varepsilon.$$

Furthermore, since $\lim_{n\to\infty} x_n = c$, there exists a positive integer N such that $n > N$ implies $|x_n - c| < \delta$. Thus for $n > N$ we have $0 < |x_n - c| < \delta$ and $x_n \in E$, so that $|f(x_n) - L| < \varepsilon$. This shows that $\lim_{n\to\infty} f(x_n) = L$.

Conversely, suppose that L is *not* a limit of f at c. Then there exists some $\varepsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in E$ such that $0 < |x-c| < \delta$ and $|f(x)-L| \geq \varepsilon$. In particular, for each $n \in \mathbb{N}$, there exists some $x_n \in E$ such that $0 < |x_n - c| < 1/n$ and $|f(x_n) - L| \geq \varepsilon$. Now the sequence $(x_n)_{n=1,2,\ldots}$ converges to c with $x_n \neq c$ for all $n \in \mathbb{N}$, but the sequence $(f(x_n))_{n=1,2,\ldots}$ does not converge to L. □

As a corollary of the above theorem, we see that a function can have at most one limit at a given point.

Let f and g be two functions from E to $\mathbb{R}$. We define the **sum** $f+g$ and the **product** fg to be the functions from E to $\mathbb{R}$ given by

$$(f+g)(x) := f(x) + g(x) \quad \text{and} \quad (fg)(x) := f(x)g(x), \quad x \in E.$$

Moreover, if $g(x) \neq 0$ for all $x \in E$, then the **quotient** f/g is the function from E to $\mathbb{R}$ defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad x \in E.$$

The following theorem can be easily proved by combining Theorem 1.1 and Theorems 2.1 and 2.2 in Chapter 2.

Theorem 1.2. *Let f and g be two functions from a subset E of $\mathbb{R}$ to $\mathbb{R}$, and let c be a limit point of E. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then*

$$\lim_{x\to c}(f+g)(x) = L + M \quad \text{and} \quad \lim_{x\to c}(fg)(x) = LM.$$

Furthermore, if $g(x) \neq 0$ for all $x \in E$ and $M \neq 0$, then

$$\lim_{x\to c}\left(\frac{f}{g}\right)(x) = \frac{L}{M}.$$

The following theorem follows from Theorem 1.1 and the squeeze theorem given in Chapter 2.

Theorem 1.3. *Suppose that E is a subset of $\mathbb{R}$, c is a limit point of E, and f, g, h are real-valued functions on E satisfying*

$$g(x) \leq f(x) \leq h(x) \quad \text{for all } x \in E.$$

If $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x) = L$.

Let f be a function from a subset E of $\mathbb{R}$ to $\mathbb{R}$ and let c be a limit point of E. We write $\lim_{x\to c} f(x) = \infty$, if for each $M > 0$ there exists some $\delta > 0$ such that

$$x \in E \text{ and } 0 < |x-c| < \delta \quad \text{imply} \quad f(x) > M.$$

We write $\lim_{x\to c} f(x) = -\infty$, if for each $M < 0$ there exists some $\delta > 0$ such that

$$x \in E \text{ and } 0 < |x - c| < \delta \quad \text{imply} \quad f(x) < M.$$

Example 1. Show that

$$\lim_{x\to 0} \frac{1}{|x|} = +\infty.$$

Proof. Let $f(x) := 1/|x|$. Then f is defined on the set $E := \mathbb{R} \setminus \{0\}$ and 0 is a limit point of E. For given $M > 0$, we choose $\delta = 1/M > 0$. Then

$$x \in E \text{ and } |x - 0| < \delta \quad \text{imply} \quad \frac{1}{|x|} > \frac{1}{\delta} = M.$$

This shows that $\lim_{x\to 0} \frac{1}{|x|} = +\infty$.

Now we consider limits at infinity. Let f be a function from a subset E of $\mathbb{R}$ to $\mathbb{R}$ such that $E \cap (a, \infty) \neq \emptyset$ for every $a \in \mathbb{R}$. We say that a real number L is a **limit of f at** ∞, and we write $\lim_{x\to\infty} f(x) = L$, if for each $\varepsilon > 0$ there exists some real number K such that

$$x \in E \text{ and } x > K \quad \text{imply} \quad |f(x) - L| < \varepsilon.$$

Similarly, let f be a function from E to $\mathbb{R}$ such that $E \cap (-\infty, b) \neq \emptyset$ for every $b \in \mathbb{R}$. We say that a real number L is a **limit of f at** $-\infty$, and we write $\lim_{x\to-\infty} f(x) = L$, if for each $\varepsilon > 0$ there exists some real number K such that

$$x \in E \text{ and } x < K \quad \text{imply} \quad |f(x) - L| < \varepsilon.$$

Analogously, we can define $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$.

Theorems 1.1, 1.2, and 1.3 can be easily extended to limits at infinity.

Example 2. Find the limit

$$\lim_{x\to\infty} \left(\sqrt{x^2 + x} - x\right).$$

Solution. Let $f(x) := \sqrt{x^2 + x} - x$ for $x > 0$. We have

$$f(x) = \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + x)}{\sqrt{x^2 + x} + x} = \frac{x}{\sqrt{x^2 + x} + x}.$$

But $x > 0$ implies $x \leq \sqrt{x^2 + x} \leq x + 1$. It follows that $2x \leq \sqrt{x^2 + x} + x \leq 2x + 1$. Hence,

$$\frac{x}{2x + 1} \leq f(x) \leq \frac{x}{2x} = \frac{1}{2}, \quad x > 0.$$

Since $\lim_{x\to\infty} x/(2x + 1) = 1/2$, by Theorem 1.3 we conclude that $\lim_{x\to\infty} f(x) = 1/2$.

§2. Continuous Functions

Let f be a function from a subset E of $\mathbb{R}$ to $\mathbb{R}$ and let $c \in E$. We say that f is **continuous at** c if for each $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$x \in E \text{ and } |x - c| < \delta \quad \text{imply} \quad |f(x) - f(c)| < \varepsilon.$$

If f is continuous at every point of a subset S of E, then f is said to be **continuous on** S. If f is continuous on its domain E, then f is said to be **continuous**.

The following theorem can be proved in a way analogous to the proof of Theorem 1.1.

Theorem 2.1. *Let f be a function from a subset E of $\mathbb{R}$ to $\mathbb{R}$ and let $c \in E$. Then f is continuous at c if and only if for every sequence $(x_n)_{n=1,2...}$ in E that converges to c, the sequence $(f(x_n))_{n=1,2,...}$ converges to $f(c)$.*

Combining Theorem 2.1 with Theorems 2.1 and 2.2 in Chapter 2, we obtain the following result.

Theorem 2.2. *Let f and g be two functions from a subset E of $\mathbb{R}$ to $\mathbb{R}$, and let $c \in E$. If f and g are continuous at c, then $f + g$ and fg are continuous at c. Furthermore, if $g(c) \neq 0$, then f/g is continuous at c.*

Example 1. Let f, g, u, v be the functions from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$f(x) := x^2, \quad g(x) := x^3, \quad u(x) := x^3 - x, \quad v(x) := \frac{x}{1 + |x|}, \quad x \in \mathbb{R}.$$

The functions f, g, u, v are all continuous on $\mathbb{R}$. Moreover, f is neither one-to-one nor onto, g is bijective, u is onto but not one-to-one, and v is one-to-one but not onto.

Let A and B be two subsets of $\mathbb{R}$. Suppose that f is a function from A to B and g is a function from B to $\mathbb{R}$. Then the **composition** $g \circ f$ is the function from A to $\mathbb{R}$ defined by

$$g \circ f(x) = g(f(x)), \quad x \in A.$$

Example 2. Let f and g be the functions from $\mathbb{R}$ to $\mathbb{R}$ given by

$$f(x) = 1 - x \quad \text{and} \quad g(x) = \frac{x}{x^2 + 1}, \quad x \in \mathbb{R}.$$

Find $g \circ f$ and $f \circ g$.

Solution. We have

$$g \circ f(x) = \frac{1 - x}{(1 - x)^2 + 1} \quad \text{and} \quad f \circ g(x) = 1 - \frac{x}{x^2 + 1}, \quad x \in \mathbb{R}.$$

Note that $f \circ g \neq g \circ f$.

Theorem 2.3. *Suppose that f is a function from A to B and g is a function from B to $\mathbb{R}$. If f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.*

Proof. Let $(x_n)_{n=1,2,\dots}$ be a sequence in A that converges to c. Since f is continuous at c, the sequence $f(x_n)$ converges to $f(c)$, by Theorem 2.1. Since g is continuous at $f(c)$, by Theorem 2.1 again we obtain

$$\lim_{n\to\infty} g \circ f(x_n) = \lim_{n\to\infty} g(f(x_n)) = g(f(c)) = g \circ f(c).$$

This is true for every sequence $(x_n)_{n=1,2,\dots}$ in A that converges to c. Therefore, $g \circ f$ is continuous at c. □

Suppose that p is a function from $\mathbb{R}$ to $\mathbb{R}$ given by

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \quad x \in \mathbb{R},$$

where $a_0, a_1, \dots, a_n \in \mathbb{R}$. Then p is called a **polynomial** function. If n is the largest integer such that $a_n \neq 0$, then we say that n is the **degree** of f. By Theorem 2.2, a polynomial function is continuous on $\mathbb{R}$.

Let p be a polynomial of degree $n \geq 1$. A real number c is said to be a **root** of p, if $p(c) = 0$. It is known that $p(c) = 0$ if and only if there exists a polynomial p_1 of degree $n-1$ such that

$$p(x) = (x - c)p_1(x), \quad x \in \mathbb{R}.$$

Consequently, a polynomial of degree n can have at most n roots.

A function r is said to be a **rational function** if $r = p/q$, where p and q are two polynomials and $q \neq 0$. Let $Z(q) := \{x \in \mathbb{R} : q(x) = 0\}$ be the set of the roots of q. Then the domain of r is the set $\mathbb{R} \setminus Z(q)$. By Theorem 2.2, a rational function is continuous on its domain. Thus, if $q(c) \neq 0$, we have

$$\lim_{x\to c} r(x) = \lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}.$$

If $q(c) = 0$ but $p(c) \neq 0$, then $\lim_{x\to c} r(x)$ does not exist. If p and q are polynomials of positive degree, and if $p(c) = 0$ and $q(c) = 0$, then there exist polynomials p_1 and q_1 such that $p(x) = (x - c)p_1(x)$ and $q(x) = (x - c)q_1(x)$ for all $x \in \mathbb{R}$. In this case we have

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \lim_{x\to c} \frac{(x - c)p_1(x)}{(x - c)q_1(x)} = \lim_{x\to c} \frac{p_1(x)}{q_1(x)}.$$

Example 3. Find the limit

$$\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4}.$$

Solution. We have

$$\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4} = \lim_{x\to 2} \frac{(x-2)(x+1)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x+1}{x+2} = \frac{3}{4}.$$

§3. Properties of Continuous Functions

Let f be a function from a set X to a set Y. If $A \subseteq X$, then $f(A)$, the **image** of A under f, is defined by

$$f(A) := \{f(x) : x \in A\}.$$

If $B \subseteq Y$, the **inverse image** of B is the set

$$f^{-1}(B) := \{x \in X : f(x) \in B\}.$$

A function f from a subset E of $\mathbb{R}$ to $\mathbb{R}$ is said to be **bounded** if the set $\{f(x) : x \in E\}$ is bounded, that is, if there exists a real number M such that $|f(x)| \le M$ for all $x \in E$.

Theorem 3.1. *Let f be a continuous function from a closed interval $[a, b]$ to $\mathbb{R}$. Then f is a bounded function. Moreover, f attains its maximum and minimum values on $[a, b]$; that is, there exist $s, t \in [a, b]$ such that $f(s) \le f(x) \le f(t)$ for all $x \in [a, b]$.*

Proof. Let $M := \sup\{f([a, b])\}$, where $f([a, b]) := \{f(x) : x \in [a, b]\}$. Note that M could be ∞ or a real number. Let $c := (a + b)/2$, $M_1 := \sup\{f([a, c])\}$ and $M_2 := \sup\{f([c, b])\}$. Then we have $M = \max\{M_1, M_2\}$. If $M_1 = M$, choose $[a_1, b_1] := [a, c]$; otherwise, choose $[a_2, b_2] := [c, b]$. Suppose that $[a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k]$ have been constructed so that $\sup\{f([a_k, b_k])\} = M$. Let $c_k := (a_k + b_k)/2$. If $\sup\{f([a_k, b_k])\} = M$, let $[a_{k+1}, b_{k+1}] := [a_k, c_k]$; otherwise, let $[a_{k+1}, b_{k+1}] := [c_k, b_k]$. By Theorem 3.2 in Chapter 2, there exists a real number t such that $a_k \le t \le b_k \,\forall\, k \in \mathbb{N}$ and $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = t$. We claim that $f(t) = M$. Suppose to the contrary that $f(t) < M$. Then there exists some $\varepsilon > 0$ such that $f(t) + \varepsilon < M$. Since f is continuous at t, there exists $\delta > 0$ such that $x \in [a, b] \cap (t - \delta, t + \delta)$ implies $f(x) < f(t) + \varepsilon$. Since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = t$, there exists a positive integer K such that $t - \delta < a_K < b_K < t + \delta$. It follows that $\sup\{f([a_K, b_K])\} \le f(t) + \varepsilon < M$. This is a contradiction. Therefore, $f(t) = M$. Thus, f is bounded above and f attains its maximum at t.

In a similar way we can prove that f is bounded below and attains its minimum at some point $s \in [a, b]$. □

The above theorem is not valid if the closed interval $[a, b]$ is replaced by an open interval.

Example 1. Let $f(x) = 1/x$, $x \in (0, 1)$. Then f is continuous but unbounded on $(0, 1)$. Moreover, we have $\inf\{f(x) : x \in (0, 1)\} = 1$. But f does not attain the value 1.

We are in a position to establish the following intermediate value theorem for continuous functions.

Theorem 3.2. *Let f be a continuous function from a closed interval $[a, b]$ to $\mathbb{R}$. If y lies between $f(a)$ and $f(b)$, that is, either $f(a) \le y \le f(b)$ or $f(b) \le y \le f(a)$, then there exists some $c \in [a, b]$ such that $f(c) = y$.*

Proof. We only deal with the case $f(a) \le y \le f(b)$; the other case can be treated similarly. We shall construct a nested sequence of closed intervals $([a_k, b_k])_{k=1,2,\ldots}$ recursively as follows. Let $a_1 := a$ and $b_1 := b$. Suppose that the intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k]$ have been constructed so that $f(a_k) \le y \le f(b_k)$. Let $c_k := (a_k + b_k)/2$. If $f(c_k) \ge y$, let $a_{k+1} := a_k$ and $b_{k+1} := c_k$; otherwise, let $a_{k+1} := c_k$ and $b_{k+1} := b_k$. Clearly, $f(a_{k+1}) \le y \le f(b_{k+1})$. In light of our construction, $[a_{k+1}, b_{k+1}] \subset [a_k, b_k]$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty}(b_k - a_k) = 0$. By Theorem 3.2 in Chapter 2, there exists a real number c such that $a_k \le c \le b_k$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = c$. Since f is continuous on $[a, b]$, we obtain

$$f(c) = \lim_{k\to\infty} f(a_k) \le y \quad \text{and} \quad f(c) = \lim_{k\to\infty} f(b_k) \ge y.$$

Therefore, $f(c) = y$. □

Example 2. Let $p(x) := x^3 - x - 1$, $x \in \mathbb{R}$. Then the cubic polynomial p has a root in $(0, 2)$.

Proof. We have $p(0) = -1 < 0 < p(2) = 5$. By the intermediate value theorem, there exists some $c \in (0, 2)$ such that $p(c) = 0$.

Let f be a continuous function from a closed interval $[a, b]$ to $\mathbb{R}$. From Theorems 3.1 and 3.2 we see that the range of f is a closed interval: $f([a, b]) = [m, M]$, where $m := \inf\{f([a, b])\}$ and $M := \sup\{f([a, b])\}$.

Theorem 3.3. *Let f be a real-valued continuous function on an interval $I \subseteq \mathbb{R}$. Then $J := f(I)$ is an interval.*

Proof. Let $m := \inf J$ and $M := \sup J$. We claim that $(m, M) \subseteq J$. Indeed, if $y \in (m, M)$, then there exist $y_1, y_2 \in J$ such that $y_1 < y < y_2$. Suppose $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

By Theorem 3.2, there exists some c between x_1 and x_2 such that $f(c) = y$. Since I is an interval, the points between x_1 and x_2 belong to I. Thus c belongs to I, and hence $y = f(c) \in f(I) = J$. This shows that $(m, M) \subseteq J$. If $m \in J$ and $M \in J$, then $J = [m, M]$; If $m \in J$ and $M \notin J$, then $J = [m, M)$; if $m \notin J$ and $M \in J$, then $J = (m, M]$; if $m \notin J$ and $M \notin J$, then $J = (m, M)$. $\square$

§4. Uniform Continuity

Let f be a function from a subset E of $\mathbb{R}$ to $\mathbb{R}$. We say that f is **uniformly continuous on** E if for each $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$x_1, x_2 \in E \text{ and } |x_1 - x_2| < \delta \quad \text{imply} \quad |f(x_1) - f(x_2)| < \varepsilon.$$

A function f from E to $\mathbb{R}$ is said to satisfy a **Lipschitz condition** on E if there exists a positive constant M such that

$$|f(x) - f(y)| \leq M|x - y| \quad \forall\, x, y \in E.$$

Clearly, if $f : E \to \mathbb{R}$ satisfies a Lipschitz condition on E, then f is uniformly continuous on E.

Example 1. Let f be the function given by $f(x) = 1/x$ for $x \in [1, \infty)$. Then f is uniformly continuous on $[1, \infty)$.

Proof. For $x, y \in [1, \infty)$ we have

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| = \frac{|x - y|}{xy} \leq |x - y|.$$

Thus f satisfies a Lipschitz condition on E, so f is uniformly continuous on $[1, \infty)$.

Theorem 4.1. *Let f be a function from a subset E of $\mathbb{R}$ to $\mathbb{R}$. If f is uniformly continuous on E, then $\lim_{n\to\infty}[f(x_n) - f(y_n)] = 0$ for any sequences $(x_n)_{n=1,2,\ldots}$ and $(y_n)_{n=1,2,\ldots}$ in E with $\lim_{n\to\infty}(x_n - y_n) = 0$.*

Proof. Suppose that f is a uniformly continuous function on E. Given $\varepsilon > 0$, there exists some $\delta > 0$ such that $x, y \in E$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \varepsilon$. Let $(x_n)_{n=1,2,\ldots}$ and $(y_n)_{n=1,2,\ldots}$ be two sequences in E such that $\lim_{n\to\infty}(x_n - y_n) = 0$. Then there exists a positive integer N such that $|x_n - y_n| < \delta$ for all $n > N$. Consequently, $|f(x_n) - f(y_n)| < \varepsilon$ for all $n > N$. This shows that $\lim_{n\to\infty}[f(x_n) - f(y_n)] = 0$. $\square$

If we wish to prove that a given function is not uniformly continuous, then we may apply the above theorem in the following way. Suppose that we can find some $\varepsilon > 0$ and two sequences $(x_n)_{n=1,2,\ldots}$ and $(y_n)_{n=1,2,\ldots}$ in E with $\lim_{n\to\infty}(x_n - y_n) = 0$ such that $|f(x_n) - f(y_n)| \ge \varepsilon$ for all $n \in \mathbb{N}$, then f is not uniformly continuous on E.

Example 2. Let g be the function given by $g(x) = 1/x$ for $x \in (0, 1]$. Then g is not uniformly continuous on $(0, 1]$.

Proof. For $n \in \mathbb{N}$, let $x_n := 1/n$ and $y_n := 1/(n + 1)$. Then $(x_n)_{n=1,2,\ldots}$ and $(y_n)_{n=1,2,\ldots}$ are sequences in $(0, 1]$ such that $\lim_{n\to\infty}(x_n-y_n) = 0$. But $|g(x_n)-g(y_n)| = |n-(n+1)| = 1$ for all $n \in \mathbb{N}$. Hence g is not uniformly continuous on $(0, 1]$.

Theorem 4.2. *If f is continuous on a bounded closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.*

Proof. Assume that f is *not* uniformly continuous on $[a, b]$. Then there exists some $\varepsilon > 0$ such that for each $\delta > 0$ the implication "$|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$ " fails. Consequently, for every $n \in \mathbb{N}$, there exist $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < 1/n$ and yet $|f(x_n) - f(y_n)| \ge \varepsilon$. By the Bolzano-Weierstrass theorem, a subsequence $(x_{n_k})_{k=1,2,\ldots}$ converges. Moreover, if $x_0 = \lim_{k\to\infty} x_{n_k}$, then x_0 belongs to $[a, b]$. Clearly we also have $x_0 = \lim_{k\to\infty} y_{n_k}$. Since f is continuous at x_0, we have

$$f(x_0) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(y_{n_k})$$

and so

$$\lim_{k\to\infty} \big[f(x_{n_k}) - f(y_{n_k})\big] = 0.$$

But $|f(x_{n_k}) - f(y_{n_k})| \ge \varepsilon$ for all k. So this is a contradiction. Hence f is uniformly continuous on $[a, b]$. □

§5. Monotone Functions and Inverse Functions

Let f be a real-valued function defined on an interval I. We say that f is **strictly increasing on I** if

$$x_1, x_2 \in I \text{ and } x_1 < x_2 \quad \text{imply} \quad f(x_1) < f(x_2),$$

strictly decreasing on I if

$$x_1, x_2 \in I \text{ and } x_1 < x_2 \quad \text{imply} \quad f(x_1) > f(x_2),$$

increasing on I if

$$x_1, x_2 \in I \text{ and } x_1 < x_2 \quad \text{imply} \quad f(x_1) \le f(x_2),$$

decreasing on I if

$$x_1, x_2 \in I \text{ and } x_1 < x_2 \quad \text{imply} \quad f(x_1) \geq f(x_2).$$

A real-valued function on I is said to be **monotone on** I if it is either increasing or decreasing. A real-valued function on I is said to be **strictly monotone on** I if it is either strictly increasing or strictly decreasing. Evidently, a strictly monotone function is one-to-one.

Example 1. Let f, g, u, v be the functions from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$f(x) := x^2, \quad g(x) := x^3, \quad u(x) := 3, \quad v(x) := -\frac{x}{1+|x|}, \quad x \in \mathbb{R}.$$

Then f is not monotone, g is strictly increasing, u is monotone but not strictly monotone, and v is strictly decreasing.

One-sided limits are often useful in the study of monotone functions. Let f be a function from a subset E of $\mathbb{R}$ to $\mathbb{R}$. Suppose that c is a limit point of $E \cap (-\infty, c)$. We say that a real number L is a **left limit of** f **at** c, and we write $\lim_{x \to c^-} f(x) = L$, if for each $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$x \in E \text{ and } c - \delta < x < c \quad \text{imply} \quad |f(x) - L| < \varepsilon.$$

Suppose that c is a limit point of $E \cap (c, \infty)$. We say that a real number L is a **right limit of** f **at** c, and we write $\lim_{x \to c^+} f(x) = L$, if for each $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$x \in E \text{ and } c < x < c + \delta \quad \text{imply} \quad |f(x) - L| < \varepsilon.$$

Let f be a function from a subset E of $\mathbb{R}$ to $\mathbb{R}$, and let c be a limit point of E. Then $\lim_{x \to c} f(x)$ exists if and only if both $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist and they are equal.

Recall that $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. We have $-\infty < a < \infty$ for every $a \in \mathbb{R}$.

Theorem 5.1. *Let $a, b, c \in \overline{\mathbb{R}}$ with $a < c < b$. If f is a monotone function from (a, b) to $\mathbb{R}$, then $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$ exist in $\overline{\mathbb{R}}$. Moreover, $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist in $\mathbb{R}$.*

Proof. Assume that f is increasing. Let $s := \sup\{f(x) : a < x < b\}$. For each $\varepsilon > 0$, $s - \varepsilon$ is not an upper bound of $f((a, b))$; there exists some $x_\varepsilon \in (a, b)$ such that $s - \varepsilon < f(x_\varepsilon) \leq s$. Since f is increasing, $s - \varepsilon < f(x) \leq s$ for all $x \in (x_\varepsilon, b)$. Therefore,

$$\lim_{x \to b^-} f(x) = s = \sup\{f(x) : a < x < b\}.$$

Similarly,

$$\lim_{x\to a^+} f(x) = \inf\{f(x) : a < x < b\}.$$

If $a < c < b$, then there exist $c_1, c_2 \in \mathbb{R}$ such that $a < c_1 < c < c_2 < b$. Let $u := \sup\{f(x) : c_1 < x < c\}$ and $v := \inf\{f(x) : c < x < c_2\}$. Then $\lim_{x\to c^-} f(x) = u$ and $\lim_{x\to c^+} f(x) = v$. But $f(c_1) \le u \le v \le f(c_2)$, so $u, v \in \mathbb{R}$. □

Theorem 5.2. *Let g be a real-valued function on an interval J in $\mathbb{R}$. If g is monotone and $I := g(J)$ is an interval, then g is continuous.*

Proof. Without loss of generality, we consider the case where g is increasing. If I is a singleton, then g is constant on J and hence g is continuous in this case. Thus we may assume that I contains at least two points.

Let $x_0 \in J$ and $y_0 = g(x_0) \in I$. We wish to prove that g is continuous at x_0.

First, suppose that y_0 is not an endpoint of I. For given $\varepsilon > 0$, there exist $y_1, y_2 \in I$ such that $y_0 - \varepsilon < y_1 < y_0 < y_2 < y_0 + \varepsilon$. Let x_1 and x_2 be the points in J such that $g(x_1) = y_1$ and $g(x_2) = y_2$. Since g is increasing, we have $x_1 < x_0 < x_2$. Choose $\delta > 0$ such that $x_1 < x_0 - \delta < x_0 < x_0 + \delta < x_2$. Consequently,

$$x_0 - \delta < x < x_0 + \delta \quad \text{implies} \quad y_0 - \varepsilon < y_1 = g(x_1) \le g(x) \le g(x_2) = y_2 < y_0 + \varepsilon.$$

This shows that g is continuous at x_0.

Second, suppose that y_0 is the left endpoint of I. For given $\varepsilon > 0$, there exists $y_2 \in I$ such that $y_0 < y_2 < y_0 + \varepsilon$. Let x_2 be the point in J such that $g(x_2) = y_2$. Since g is increasing, we have $x_0 < x_2$. Choose $\delta > 0$ such that $x_0 < x_0 + \delta < x_2$. Consequently,

$$x_0 - \delta < x < x_0 + \delta \text{ and } x \in J \implies y_0 \le g(x) \le g(x_2) = y_2 < y_0 + \varepsilon.$$

This shows that g is continuous at x_0.

Third, suppose that y_0 is the right endpoint of I. A similar argument shows that g is continuous at x_0.

Since g is continuous at every point in J, we conclude that g is continuous on J. □

Theorem 5.3. *Let f be a function from an interval I to $\mathbb{R}$. If f is strictly increasing (decreasing), then so is its inverse function f^{-1}. If, in addition, f is continuous, then so is f^{-1}.*

Proof. Suppose that f is a strictly increasing function from an interval I to $\mathbb{R}$. Let $g := f^{-1}$. Then $g(y) = x$ if and only if $f(x) = y$. Suppose that $x_1 = g(y_1)$ and $x_2 = g(y_2)$, where $y_1, y_2 \in J := f(I)$. If $y_1 < y_2$, we must have $x_1 < x_2$, for otherwise $x_1 \geq x_2$ would imply $y_1 = f(x_1) \geq f(x_2) = y_2$. This shows that g is strictly increasing.

If, in addition, f is continuous, then $J = f(I)$ is an interval by Theorem 3.2. Now g is a monotone function from the interval J onto the interval I. By Theorem 5.2 we conclude that g is continuous. $\square$

Let us apply the above theorem to the function f_n given by $f_n(x) = x^n$ for $x \in \mathbb{R}$, where $n \in \mathbb{N}$. Evidently, f_n is a continuous function on $\mathbb{R}$. If n is an odd integer, then f_n is a strictly increasing function on $\mathbb{R}$ and f_n maps $\mathbb{R}$ onto $\mathbb{R}$. Hence, for any $b \in \mathbb{R}$, there exists a unique $a \in \mathbb{R}$ such that $a^n = b$. If n is an even integer, then f_n is a strictly increasing function on $[0, \infty)$ and f_n maps $[0, \infty)$ onto $[0, \infty)$. Hence, for any $b \in [0, \infty)$, there exists a unique $a \in [0, \infty)$ such that $a^n = b$. In both cases, we call a the nth root of b and write $a = \sqrt[n]{b}$. If n is an odd integer, then the root function $g_n : x \mapsto \sqrt[n]{x}$ is a continuous and strictly increasing function from $\mathbb{R}$ onto $\mathbb{R}$. If n is an even integer, then the root function $g_n : x \mapsto \sqrt[n]{x}$ is a continuous and strictly increasing function from $[0, \infty)$ onto $[0, \infty)$.

§6. The Exponential and Logarithmic Functions

For $a > 0$, let f_a be the exponential function on $\mathbb{R}$ given by $f_a(x) := a^x$, $x \in \mathbb{R}$. If $(\alpha_n)_{n=1,2,\ldots}$ is a sequence of rational numbers such that $\lim_{n\to\infty} \alpha_n = x$, then

$$\lim_{n\to\infty} a^{\alpha_n} = a^x.$$

Moreover, for $x, y \in \mathbb{R}$ we have

$$a^x a^y = a^{x+y}, \ a^x/a^y = a^{x-y}, \text{ and } (a^x)^y = a^{xy}.$$

We claim that, for $a > 1$, the function f_a is strictly increasing on $(-\infty, \infty)$. Indeed, if $-\infty < x < y < \infty$, then there exist rational numbers r and s such that $x < r < s < y$. We can find two sequences $(\alpha_n)_{n=1,2,\ldots}$ and $(\beta_n)_{n=1,2,\ldots}$ of rational numbers such that $\lim_{n\to\infty} \alpha_n = x$, $\lim_{n\to\infty} \beta_n = y$, and that $\alpha_n \leq r < s \leq \beta_n$ for all $n \in \mathbb{N}$. It follows that

$$a^{\alpha_n} \leq a^r < a^s \leq a^{\beta_n} \quad \forall\, n \in \mathbb{N}.$$

Letting n go to ∞ in the above inequalities, we obtain

$$a^x \leq a^r < a^s \leq a^y.$$

This justifies our claim. If $a = 1$, f_a is the constant function 1. If $0 < a < 1$, then $f_a(x) = a^x = (1/a)^{-x}$ with $1/a > 1$. Hence, for $a \in (0,1)$, the function f_a is strictly decreasing on $(-\infty, \infty)$.

Fix $a > 1$. We have $\lim_{n\to\infty} a^n = \infty$. Thus, given $M > 0$, there exists a positive integer N such that $a^N > M$. Consequently, $x > N$ implies $a^x > a^N > M$. This shows that $\lim_{x\to\infty} a^x = \infty$. It follows that

$$\lim_{x\to-\infty} a^x = \lim_{y\to\infty} a^{-y} = \lim_{y\to\infty} \frac{1}{a^y} = 0.$$

If $0 < a < 1$, then

$$\lim_{x\to\infty} a^x = \lim_{x\to\infty} (1/a)^{-x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} a^x = \lim_{x\to-\infty} (1/a)^{-x} = \infty.$$

Theorem 6.1. *For $a > 0$, the exponential function $f_a : x \mapsto a^x$ is a continuous function on $\mathbb{R}$. If $a > 1$, f_a is a strictly increasing function from $(-\infty, \infty)$ onto $(0, \infty)$. If $0 < a < 1$, f_a is a strictly decreasing function from $(-\infty, \infty)$ onto $(0, \infty)$.*

Proof. Fix $a > 1$. Let $\varepsilon > 0$ be given. Since $\lim_{k\to\infty} a^{1/k} = a^{-1/k} = 1$, there exists a positive integer K such that $1 - \varepsilon < a^{-1/K} < a^{1/K} < 1 + \varepsilon$. Choose $\delta := 1/K$. Then $-\delta < x < \delta$ implies $a^{-1/K} < a^x < a^{1/K}$, and hence $1 - \varepsilon < a^x < 1 + \varepsilon$. This shows that $\lim_{x\to 0} a^x = 1$. Now let α be an arbitrary real number. Then

$$\lim_{x\to\alpha} a^x = \lim_{x\to\alpha} \left(a^\alpha a^{x-\alpha}\right) = a^\alpha \lim_{x\to\alpha} a^{x-\alpha} = a^\alpha \lim_{x-\alpha\to 0} a^{x-\alpha} = a^\alpha \cdot 1 = a^\alpha.$$

Thus, f_a is continuous at every point $\alpha \in \mathbb{R}$. If $a = 1$, then f_a is the constant function 1. If $0 < a < 1$, then $a^x = 1/(1/a)^{-x}$. So f_a is also continuous on $\mathbb{R}$.

If $a > 1$, then f_a is strictly increasing on $(-\infty, \infty)$. Moreover, $\lim_{x\to-\infty} a^x = 0$ and $\lim_{x\to\infty} a^x = \infty$. By the proof of Theorem 3.3 we conclude that the range of f_a is the interval $(0, \infty)$. If $0 < a < 1$, then f_a is strictly decreasing on $(-\infty, \infty)$. Moreover, $\lim_{x\to-\infty} a^x = \infty$ and $\lim_{x\to\infty} a^x = 0$. So the range of f_a is also the interval $(0, \infty)$. $\square$

The above theorem tells us that, for $a \in (0,1) \cup (1, \infty)$, f_a is a bijective function from $(-\infty, \infty)$ to $(0, \infty)$. Consequently, for given $\beta \in (0, \infty)$, there exists a unique $\alpha \in \mathbb{R}$ such that $a^\alpha = \beta$. We write $\alpha = \log_a \beta$ and call α the **logarithm** of α to base a. Let $g_a(x) := \log_a x$ for $x \in (0, \infty)$. Then g_a is the inverse function of f_a. We call g_a the **logarithmic function** to base a. The following identities follow from the definition at once:

$$a^{\log_a y} = y \quad \forall\, y \in (0, \infty) \qquad \text{and} \qquad \log_a(a^x) = x \quad \forall\, x \in (-\infty, \infty).$$

By Theorem 5.3 and Theorem 6.1 we have the following result.

Theorem 6.2. *For $a \in (0,1) \cup (1,\infty)$, the logarithmic function $g_a : x \mapsto \log_a x$ is a continuous function on $(0,\infty)$. If $a > 1$, g_a is a strictly increasing function from $(0,\infty)$ onto $(-\infty,\infty)$. If $0 < a < 1$, g_a is a strictly decreasing function from $(0,\infty)$ onto $(-\infty,\infty)$.*

Let us study some properties of the logarithmic function. For $a > 1$ we have

$$\lim_{x\to 0^+} \log_a x = -\infty \quad \text{and} \quad \lim_{x\to\infty} \log_a x = \infty.$$

Suppose $x, y > 0$, $u = \log_a x$, and $v = \log_a y$. Then we have $x = a^u$ and $y = a^v$. It follows that

$$\log_a(xy) = \log_a(a^u a^v) = \log_a(a^{u+v}) = u + v = \log_a x + \log_a y$$

and

$$\log_a(x/y) = \log_a(a^u/a^v) = \log_a(a^{u-v}) = u - v = \log_a x - \log_a y.$$

Moreover, for $\mu \in \mathbb{R}$ we have

$$\log_a(x^\mu) = \log_a(a^u)^\mu = \log_a a^{u\mu} = \mu u = \mu \log_a x.$$

Suppose that $a, b \in (0,1) \cup (1,\infty)$. For $x \in (0,\infty)$, let $\mu := \log_b x$. Then $x = b^\mu$ and

$$\log_a x = \log_a b^\mu = \mu \log_a b = (\log_b x)(\log_a b).$$

This leads to the following formula for change of bases:

$$\log_b x = \frac{\log_a x}{\log_a b} \quad \forall\, x \in (0,\infty).$$

Fix $\mu \in \mathbb{R}$. Let h_μ be the power function given by

$$h_\mu(x) := x^\mu, \quad x \in (0,\infty).$$

Since $x = 2^{\log_2 x}$ for $x > 0$, we have

$$x^\mu = \left(2^{\log_2 x}\right)^\mu = 2^{\mu \log_2 x}, \quad x \in (0,\infty).$$

Recall that the composition of two continuous functions is continuous. So the power function $h_\mu : x \mapsto x^\mu$ is continuous on its domain $(0,\infty)$. Its range is also $(0,\infty)$. If $\mu > 0$, the function $x \mapsto \mu \log_2 x$ is strictly increasing on $(0,\infty)$; hence $h_\mu : x \mapsto x^\mu$ is a strictly increasing function on $(0,\infty)$. Moreover,

$$\lim_{x\to 0^+} x^\mu = 0 \quad \text{and} \quad \lim_{x\to\infty} x^\mu = \infty.$$

If $\mu < 0$, $h_\mu : x \mapsto x^\mu$ is a strictly decreasing function on $(0,\infty)$. Further,

$$\lim_{x\to 0^+} x^\mu = \infty \quad \text{and} \quad \lim_{x\to\infty} x^\mu = 0.$$