Chapter 2. Sequences

§1. Limits of Sequences

Let A be a nonempty set. A function from $\mathbb{N}$ to A is called a **sequence** of elements in A. We often use $(a_n)_{n=1,2,\ldots}$ to denote a sequence. By this we mean that a function f from $\mathbb{N}$ to some set A is given and $f(n) = a_n \in A$ for $n \in \mathbb{N}$. More generally, a function from a subset of $\mathbb{Z}$ to A is also called a sequence.

It is important to distinguish between a sequence and its set of values. The sequence $(a_n)_{n=1,2,\ldots}$ given by $a_n = (-1)^n$ for $n \in \mathbb{N}$ has infinitely many terms even though their values are repeated over and over. On the other hand, the *set* $\{(-1)^n : n \in \mathbb{N}\}$ is exactly the set $\{-1, 1\}$ consisting of two numbers.

A sequence $(a_n)_{n=1,2,\ldots}$ of real numbers is said to **converge** to the real number a provided that for each $\varepsilon > 0$ there exists a positive integer N such that $|a_n - a| < \varepsilon$ whenever $n > N$. If $(a_n)_{n=1,2,\ldots}$ converges to a, we write $\lim_{n\to\infty} a_n = a$. The number a is called the **limit** of the sequence $(a_n)_{n=1,2,\ldots}$. A sequence that does not converge to some real number is said to **diverge**.

Example 1. Prove that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Proof. For given $\varepsilon > 0$, we wish to find a positive integer N such that $n > N$ implies $|\frac{1}{n} - 0| < \varepsilon$. The latter is equivalent to $n > 1/\varepsilon$. Choose $N = \lfloor 1/\varepsilon \rfloor + 1$. If $n > N$, then $n > 1/\varepsilon$, and hence $|\frac{1}{n} - 0| < \varepsilon$. This shows that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example 2. If $|r| < 1$, then

$$\lim_{n\to\infty} r^n = 0.$$

Proof. If $r = 0$, then $r^n = 0$ for all $n \in \mathbb{N}$. Obviously, $\lim_{n\to\infty} r^n = 0$ in this case. Suppose $r \neq 0$. Since $|r| < 1$, we have $1/|r| > 1$. Let $b := 1/|r| - 1$. Then $b > 0$ and $1/|r| = 1 + b$. It follows that $|r| = 1/(1+b)$ and $|r^n| = 1/(1+b)^n$. By the Bernoulli inequality, $(1+b)^n \geq 1 + nb$ for all $n \in \mathbb{N}$. Consequently,

$$|r|^n = \frac{1}{(1+b)^n} \leq \frac{1}{1+nb} < \frac{1}{nb}.$$

For given $\varepsilon > 0$, we wish to find a positive integer N such that $n > N$ implies $|r^n| < \varepsilon$. This happens if $\frac{1}{nb} < \varepsilon$, *i.e.*, $n > 1/(b\varepsilon)$. Choose $N := \lfloor 1/(b\varepsilon) \rfloor + 1$. If $n > N$, we have $n > 1/(b\varepsilon)$, and hence $|r^n| < \varepsilon$. This shows $\lim_{n\to\infty} r^n = 0$.

Theorem 1.1. *A convergent sequence of real numbers has a unique limit.*

Proof. Let $(a_n)_{n=1,2,\ldots}$ be a convergent sequence. Suppose that $\lim_{n\to\infty} a_n = s$ and $\lim_{n\to\infty} a_n = t$. We wish to prove $s = t$. For given $\varepsilon > 0$, by the definition of limit, there exists a positive integer N_1 such that

$$n > N_1 \quad \text{implies} \quad |a_n - s| < \varepsilon/2.$$

Moreover, there exists a positive integer N_2 such that

$$n > N_2 \quad \text{implies} \quad |a_n - t| < \varepsilon/2.$$

For $n > \max\{N_1, N_2\}$, by the triangle inequality we have

$$|s - t| = |(s - a_n) + (a_n - t)| \leq |a_n - s| + |a_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $|s - t| < \varepsilon$ for all $\varepsilon > 0$. It follows that $|s - t| = 0$ and hence $s = t$. □

A sequence $(a_n)_{n=1,2,\ldots}$ of real numbers is said to be **bounded** if the set $\{a_n : n \in \mathbb{N}\}$ is bounded in $\mathbb{R}$.

Theorem 1.2. *A convergent sequence of real numbers is bounded.*

Proof. Let $(a_n)_{n=1,2,\ldots}$ be a convergent sequence such that $\lim_{n\to\infty} a_n = a$. For $\varepsilon = 1$ there exists a positive integer N such that

$$n > N \quad \text{implies} \quad |a_n - a| < 1.$$

For $n > N$, it follows that

$$|a_n| = |a + (a_n - a)| \leq |a| + |a_n - a| < |a| + 1.$$

Define $M := \max\{|a_1|, \ldots, |a_N|, |a| + 1\}$. Then we have $|a_n| \leq M$ for all $n \in \mathbb{N}$. Hence, $(a_n)_{n=1,2,\ldots}$ is a bounded sequence. □

A sequence $(a_n)_{n=1,2,\ldots}$ of real numbers is said to **diverge** to $+\infty$ provided that for each $M > 0$ there exists a positive integer N such that $a_n > M$ whenever $n > N$. In this case we write $\lim_{n\to\infty} a_n = +\infty$. Similarly, we say that $(a_n)_{n=1,2,\ldots}$ **diverges** to $-\infty$ and write $\lim_{n\to\infty} a_n = -\infty$ provided for each $M < 0$ there exists a positive integer N such that $a_n < M$ whenever $n > N$.

It is important to note that the symbols $+\infty$ and $-\infty$ do not represent real numbers. When $\lim_{n\to\infty} a_n = +\infty$ (or $-\infty$), we shall say that the limit exists, but this does not mean that the sequence converges; in fact, it diverges.

Theorem 1.3. *For a sequence* $(a_n)_{n=1,2,\ldots}$ *of positive real numbers,* $\lim_{n\to\infty} a_n = +\infty$ *holds if and only if* $\lim_{n\to\infty}(1/a_n) = 0$.

Proof. Suppose $\lim_{n\to\infty} a_n = +\infty$. Given $\varepsilon > 0$, let $M := 1/\varepsilon$. Since $\lim_{n\to\infty} a_n = +\infty$, there exists a positive integer N such that $n > N$ implies $a_n > M = 1/\varepsilon$. Consequently, $n > N$ implies $\left|\frac{1}{a_n} - 0\right| < \varepsilon$. This shows that $\lim_{n\to\infty}(1/a_n) = 0$.

Suppose $\lim_{n\to\infty}(1/a_n) = 0$. For $M > 0$, let $\varepsilon := 1/M$. Since $\lim_{n\to\infty}(1/a_n) = 0$, there exists a positive integer N such that $n > N$ implies $\frac{1}{a_n} < \varepsilon$. Consequently, $n > N$ implies $a_n > 1/\varepsilon = M$. This shows that $\lim_{n\to\infty} a_n = +\infty$. $\square$

Example 3. If $|r| > 1$, then the sequence $(r^n)_{n=1,2,\ldots}$ is unbounded.
Proof. Since $|r| > 1$, we have $1/|r| < 1$, and hence

$$\lim_{n\to\infty} \frac{1}{|r|^n} = \lim_{n\to\infty} \left(\frac{1}{|r|}\right)^n = 0.$$

By Theorem 1.3, it follows that $\lim_{n\to\infty} |r|^n = \infty$. Therefore, for $|r| > 1$, the sequence $(r^n)_{n=1,2,\ldots}$ is unbounded.

§2. Limit Theorems for Sequences

In this section we will investigate some of the important properties of sequences of real numbers. We start with algebraic operations on convergent sequences.

Theorem 2.1. *If* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$, *then*

$$\lim_{n\to\infty} (a_n + b_n) = a + b \quad \textit{and} \quad \lim_{n\to\infty} (a_n - b_n) = a - b.$$

Proof. For given $\varepsilon > 0$, there exists a positive integer N_1 such that

$$n > N_1 \quad \text{implies} \quad |a_n - a| < \varepsilon/2.$$

Moreover, there exists a positive integer N_2 such that

$$n > N_2 \quad \text{implies} \quad |b_n - b| < \varepsilon/2.$$

Let $N := \max\{N_1, N_2\}$. If $n > N$, then by the triangle inequality we have

$$|(a_n \pm b_n) - (a \pm b)| \le |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof. $\square$

If $(a_n)_{n=1,2,\ldots}$ is a convergent sequence, then the above theorem tells us that

$$\lim_{n\to\infty} (a_{n+1} - a_n) = \lim_{n\to\infty} a_{n+1} - \lim_{n\to\infty} a_n = 0.$$

Example 1. Let $a_n := (-1)^n$ for $n \in \mathbb{N}$. The sequence $(a_n)_{n=1,2,\ldots}$ diverges.
Proof. We have $|a_{n+1} - a_n| = 2$ for all $n \in \mathbb{N}$. So the sequence $(a_n)_{n=1,2,\ldots}$ diverges.

Theorem 2.2. *If* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$, *then*

$$\lim_{n\to\infty} (a_n b_n) = ab.$$

Moreover, if $b_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $b \neq 0$, *then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Proof. We have

$$|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| \leq |a_n b_n - a_n b| + |a_n b - ab| = |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a|.$$

By Theorem 1.2, there exists a real number $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. The number M can be so chosen that $|b| \leq M$. For given $\varepsilon > 0$, since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, there exists a positive integer N such that $n > N$ implies

$$|a_n - a| < \frac{\varepsilon}{2M} \quad \text{and} \quad |b_n - b| < \frac{\varepsilon}{2M}.$$

Consequently, if $n > N$, then

$$|a_n b_n - ab| \leq |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a| \leq M|b_n - b| + M|a_n - a| < \varepsilon.$$

This shows that $\lim_{n\to\infty}(a_n b_n) = ab$.

To handle quotients of sequences, we first deal with reciprocals. We begin by considering the equality

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_n b}\right| = \frac{|b - b_n|}{|b_n| \cdot |b|}.$$

For given $\varepsilon > 0$, since $\lim_{n\to\infty} b_n = b \neq 0$, there exists a positive integer N_1 such that

$$n > N_1 \quad \text{implies} \quad |b_n - b| < |b|/2.$$

It follows that $|b| = |b_n + (b - b_n)| \leq |b_n| + |b - b_n| < |b_n| + |b|/2$. Consequently, for $n > N_1$ we have $|b_n| > |b|/2$ and

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b_n| \cdot |b|} \leq \frac{|b - b_n|}{|b|^2/2}.$$

There exists a positive integer $N > N_1$ such that

$$n > N \quad \text{implies} \quad |b_n - b| < \varepsilon |b|^2/2.$$

Hence, for $n > N$ we have

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| \leq \frac{|b - b_n|}{|b|^2/2} < \varepsilon.$$

This shows that $\lim_{n\to\infty}(1/b_n) = 1/b$. Since $\lim_{n\to\infty} a_n = a$, by the first part of the theorem we obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = \lim_{n\to\infty} a_n \lim_{n\to\infty} \frac{1}{b_n} = a \cdot \frac{1}{b} = \frac{a}{b}.$$

This completes the proof. □

Example 2. Find $\lim_{n\to\infty} a_n$, where

$$a_n := \frac{n^3 + 6n^2 + 7}{4n^3 + 3n - 4}, \quad n \in \mathbb{N}.$$

Solution. We have

$$a_n = \frac{n^3\left(1 + \frac{6}{n} + \frac{7}{n^3}\right)}{n^3\left(4 + \frac{3}{n^2} - \frac{4}{n^3}\right)} = \frac{1 + \frac{6}{n} + \frac{7}{n^3}}{4 + \frac{3}{n^2} - \frac{4}{n^3}}.$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, by Theorem 2.2 we have

$$\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n^3} = \lim_{n\to\infty} \left(\frac{1}{n^2}\right)\left(\frac{1}{n}\right) = 0.$$

By Theorems 2.1 and 2.2, it follows that

$$\lim_{n\to\infty} \left(1 + \frac{6}{n} + \frac{7}{n^3}\right) = 1 \quad \text{and} \quad \lim_{n\to\infty} \left(4 + \frac{3}{n^2} - \frac{4}{n^3}\right) = 4.$$

Applying Theorem 2.2 again, we obtain

$$\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty}\left(1 + \frac{6}{n} + \frac{7}{n^3}\right)}{\lim_{n\to\infty}\left(4 + \frac{3}{n^2} - \frac{4}{n^3}\right)} = \frac{1}{4}.$$

Theorem 2.3. *Suppose that* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *If there is some* $n_0 \in \mathbb{N}$ *such that* $a_n \le b_n$ *for all* $n \ge n_0$, *then* $a \le b$.

Proof. Suppose that $a > b$. Let $\varepsilon := (a - b)/2 > 0$. Since $\lim_{n\to\infty} a_n = a$, there exists a positive integer N_1 such that

$$n > N_1 \quad \text{implies} \quad a - \varepsilon < a_n < a + \varepsilon.$$

Since $\lim_{n\to\infty} b_n = b$, there exists a positive integer N_2 such that

$$n > N_2 \quad \text{implies} \quad b - \varepsilon < b_n < b + \varepsilon.$$

Let $N := \max\{N_1, N_2, n_0\}$. Then for $n > N$ we have

$$b_n < b + \varepsilon = a - \varepsilon < a_n,$$

which contradicts the assumption that $a_n \le b_n$ for all $n \ge n_0$. Thus we conclude that $a \le b$. □

Example 3. Let $a_n := 0$ and $b_n := 1/n$ for $n \in \mathbb{N}$. Then $a_n < b_n$ for all $n \in \mathbb{N}$. But $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$. So we do not have a strict inequality for the limits.

Theorem 2.4. *Let* $(a_n)_{n=1,2,\ldots}$, $(b_n)_{n=1,2,\ldots}$, *and* $(x_n)_{n=1,2,\ldots}$ *be three sequences of real numbers. Suppose that* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s$. *If there exists some* $n_0 \in \mathbb{N}$ *such that* $a_n \le x_n \le b_n$ *for all* $n \ge n_0$, *then*

$$\lim_{n\to\infty} x_n = s.$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = s$, there exists a positive integer N_1 such that

$$n > N_1 \quad \text{implies} \quad s - \varepsilon < a_n < s + \varepsilon.$$

Since $\lim_{n\to\infty} b_n = s$, there exists a positive integer N_2 such that

$$n > N_2 \quad \text{implies} \quad s - \varepsilon < b_n < s + \varepsilon.$$

Let $N := \max\{N_1, N_2, n_0\}$. Then for $n > N$ we have $a_n \le x_n \le b_n$ and hance

$$s - \varepsilon < x_n < s + \varepsilon.$$

This shows that $\lim_{n\to\infty} x_n = s$. □

The above theorem is often called the *squeeze theorem*. The following two examples illustrate applications of the theorem.

Example 4. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
Proof. Note that $-|a_n| \le a_n \le |a_n|$. Since $\lim_{n\to\infty} |a_n| = 0$, we have $\lim_{n\to\infty} -|a_n| = 0$. By Theorem 2.4 we conclude that $\lim_{n\to\infty} a_n = 0$. In particular, for $a_n = (-1)^n/n$, we obtain $\lim_{n\to\infty}(-1)^n/n = 0$.

Example 5. If $a > 0$, then $\lim_{n\to\infty} a^{1/n} = 1$.
Proof. First, consider the case $a \ge 1$. In this case, we have

$$1 \le a^{1/n} \le 1 + \frac{a-1}{n} \quad \forall n \in \mathbb{N}.$$

The first inequality is valid because $1^n = 1 \le a$. The second inequality comes from the Bernoulli inequality. Indeed, since $(a-1)/n \ge 0$, the Bernoulli inequality gives

$$\left(1 + \frac{a-1}{n}\right)^n \ge 1 + n\frac{a-1}{n} = 1 + (a-1) = a.$$

Since $\lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty}\big(1 + (a-1)/n\big) = 1$, by the squeeze theorem we obtain $\lim_{n\to\infty} a^{1/n} = 1$.

It remains to deal with the case $0 < a < 1$. In this case, $1/a > 1$. By what has been proved, $\lim_{n\to\infty}(1/a)^{1/n} = 1$. Therefore, $\lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} 1/(1/a)^{1/n} = 1$.

More generally, if $a > 0$ and if $(\alpha_n)_{n=1,2,\ldots}$ is a sequence of rational numbers such that $\lim_{n\to\infty} \alpha_n = 0$, then $\lim_{n\to\infty} a^{\alpha_n} = 1$. To prove this result, we first consider the case $a > 1$. Since $\lim_{k\to\infty} a^{1/k} = 1$ and $\lim_{k\to\infty} a^{-1/k} = 1$, for any given $\varepsilon > 0$, there exists some $K \in \mathbb{N}$ such that

$$1 - \varepsilon < a^{-1/K} < a^{1/K} < 1 + \varepsilon.$$

But $\lim_{n\to\infty} \alpha_n = 0$. Hence, there exists some $n_0 \in \mathbb{N}$ such that $-1/K < \alpha_n < 1/K$ whenever $n \geq n_0$. Thus, for $n \geq n_0$ we have

$$a^{-1/K} < a^{\alpha_n} < a^{1/K}.$$

Consequently, $1 - \varepsilon < a^{\alpha_n} < 1 + \varepsilon$ whenever $n \geq n_0$. This proves $\lim_{n\to\infty} a^{\alpha_n} = 1$. The proof for the case $a = 1$ is trivial. It remains to consider the case $0 < a < 1$. In this case, we have $1/a > 1$ and $\lim_{n\to\infty} a^{\alpha_n} = \lim_{n\to\infty}(1/a)^{-\alpha_n} = 1$. This completes the proof.

Theorem 2.5. *Let $(a_n)_{n=1,2,\ldots}$ and $(b_n)_{n=1,2,\ldots}$ be two sequences of real numbers. If $\lim_{n\to\infty} a_n = +\infty$ and $\lim_{n\to\infty} b_n = b > 0$, then $\lim_{n\to\infty}(a_n b_n) = +\infty$.*

Proof. Select a real number m so that $0 < m < b$. Since $\lim_{n\to\infty} b_n = b > m$, there exists a positive integer N_1 such that

$$n > N_1 \quad \text{implies} \quad b_n > m.$$

Let $M > 0$. Since $\lim_{n\to\infty} a_n = +\infty$, there exists a positive integer N_2 such that

$$n > N_2 \quad \text{implies} \quad a_n > \frac{M}{m}.$$

Put $N := \max\{N_1, N_2\}$. Then $n > N$ implies $a_n b_n > (M/m)\cdot m = M$. This shows that $\lim_{n\to\infty}(a_n b_n) = +\infty$. □

Example 5. Find $\lim_{n\to\infty} a_n$ if

$$a_n = \frac{n^2 - 3}{n+1}, \quad n \in \mathbb{N}.$$

Solution. We have

$$a_n = \frac{n^2 - 3}{n+1} = \frac{n^2(1 - \frac{3}{n^2})}{n(1 + \frac{1}{n})} = n\cdot\frac{1 - \frac{3}{n^2}}{1 + \frac{1}{n}}.$$

Since $\lim_{n\to\infty} n = +\infty$ and $\lim_{n\to\infty}(1 - \frac{3}{n^2})/(1 + \frac{1}{n}) = 1$, by Theorem 2.5 we conclude that $\lim_{n\to\infty} a_n = +\infty$.

§3. Monotone Sequences

A sequence $(a_n)_{n=1,2,\ldots}$ is called an **increasing sequence** if $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$. It is called a **decreasing sequence** if $a_n \ge a_{n+1}$ for all $n \in \mathbb{N}$. A sequence $(a_n)_{n=1,2,\ldots}$ is said to be a **monotone sequence** if it is either increasing or decreasing.

Example 1. For $n \in \mathbb{N}$, let $a_n := n^3$, $b_n := 1 - 1/n$, $c_n := 1/n^2$, and $d_n := (-1)^n$. Then the sequences $(a_n)_{n=1,2,\ldots}$ and $(b_n)_{n=1,2,\ldots}$ are increasing, the sequence $(c_n)_{n=1,2,\ldots}$ is decreasing, but the sequence $(d_n)_{n=1,2,\ldots}$ is not monotone.

Theorem 3.1. *Every bounded monotone sequence of real numbers converges.*

Proof. Suppose that $(a_n)_{n=1,2,\ldots}$ is a bounded increasing sequence. By S we denote the set $\{a_n : n \in \mathbb{N}\}$ and let $u := \sup S$. Since S is bounded, u represents a real number. Given $\varepsilon > 0$, $u - \varepsilon$ is not an upper bound for S; hence there exists some $N \in \mathbb{N}$ such that $a_N > u - \varepsilon$. Since $(a_n)_{n=1,2,\ldots}$ is an increasing sequence, we have $a_N \le a_n$ for all $n > N$. Thus $n > N$ implies $u - \varepsilon < a_n \le u$. This proves that $\lim_{n\to\infty} a_n = u$. An analogous argument shows that every bounded decreasing sequence converges. □

Example 2. Let $(a_n)_{n=1,2,\ldots}$ be a sequence of positive real numbers. If

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = t < 1,$$

then $\lim_{n\to\infty} a_n = 0$.

Proof. Choose a real number q such that $t < q < 1$. Let $b_n := a_{n+1}/a_n$ for $n \in \mathbb{N}$. Since $\lim_{n\to\infty} b_n = t$, there exists a positive integer N such that $b_n < q$ for all $n \ge N$. We have $a_{n+1} = a_n b_n$ and hence $a_{n+1} \le a_n q \le a_n$ for $n \ge N$. Thus the sequence $(a_n)_{n=1,2,\ldots}$ is decreasing starting from the Nth term. By Theorem 3.1, the sequence converges. Let $s := \lim_{n\to\infty} a_n = s$. It follows from $a_{n+1} = a_n b_n$ that

$$s = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (a_n b_n) = st.$$

Consequently, $s(1-t) = 0$. But $1 - t > 0$. Therefore, $s = 0$ as desired. □

Example 3. For a real number c,

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0.$$

Proof. The assertion is obviously true for $c = 0$. Let us consider the case $c > 0$. For $n \in \mathbb{N}$, let $a_n := c^n/n!$ and $b_n := a_{n+1}/a_n$. Then

$$b_n = \frac{c^{n+1}}{(n+1)!} \frac{n!}{c^n} = \frac{c^{n+1}}{c^n} \frac{n!}{(n+1)!} = \frac{c}{n+1}.$$

It follows that $\lim_{n\to\infty} b_n = 0$. By Example 2, we have $\lim_{n\to\infty} a_n = 0$. If $c < 0$, we have $|a_n| = |c|^n/n!$. By what has been proved, $\lim_{n\to\infty} |a_n| = 0$. Consequently, $\lim_{n\to\infty} a_n = 0$.

As an application of Theorem 3.1 we prove the following result, usually referred to as the property of nested intervals. Note that a bounded closed interval is represented by $[a, b]$, where $a \le b$. Its length is $|I| := b - a$.

Theorem 3.2. *If $(I_n)_{n=1,2,\ldots}$ is a sequence of closed and bounded intervals such that $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$, then $\cap_{n=1}^{\infty} I_n$ is nonempty. If, in addition, $\lim_{n\to\infty} |I_n| = 0$, then $\cap_{n=1}^{\infty} I_n = \{c\}$ for some real number c.*

Proof. Suppose that $I_n = [a_n, b_n]$, $a_n, b_n \in \mathbb{R}$, and $a_n \le b_n$. Since $I_{n+1} \subseteq I_n$, we have $a_n \le a_{n+1}$ and $b_{n+1} \le b_n$ for all $n \in \mathbb{N}$. Thus, $(a_n)_{n=1,2,\ldots}$ is an increasing sequence and $(b_n)_{n=1,2,\ldots}$ is a decreasing sequence. We have $a_n \le b_1$ and $b_n \ge a_1$ for all $n \in \mathbb{N}$. Hence, the sequences $(a_n)_{n=1,2,\ldots}$ and $(b_n)_{n=1,2,\ldots}$ are bounded. By Theorem 3.1, $(a_n)_{n=1,2,\ldots}$ converges to some real number a. Similarly, $(b_n)_{n=1,2,\ldots}$ converges to some real number b. We have $a_n \le a \le b \le b_n$ for all $n \in \mathbb{N}$. It follows that $[a, b] \subseteq I_n$ for all $n \in \mathbb{N}$. Moreover, if $x \in \cap_{n=1}^{\infty} I_n$, then $a_n \le x \le b_n$. Consequently

$$a = \lim_{n\to\infty} a_n \le x \le \lim_{n\to\infty} b_n = b.$$

Hence $\cap_{n=1}^{\infty} I_n = [a, b]$. We have $b - a \le |I_n|$ for all $n \in \mathbb{N}$. If, in addition, $\lim_{n\to\infty} |I_n| = 0$, then $b - a = 0$. In this case $\cap_{n=1}^{\infty} I_n$ consists of only one real number. □

Example 4. The above theorem does not hold for open intervals. Set $I_n := (0, 1/n)$ for $n \in \mathbb{N}$. Then $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. But $\cap_{n=1}^{\infty} I_n = \emptyset$.

§4. Subsequences and Cauchy Sequences

Suppose that $(a_n)_{n=1,2,\ldots}$ is a sequence of real numbers. A **subsequence** of this sequence is a sequence of the form $(b_k)_{k=1,2,\ldots}$, where for each k there is a positive integer n_k such that $b_k = a_{n_k}$ for $k \in \mathbb{N}$ and that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots.$$

If $\lim_{n\to\infty} a_n = c$, then every subsequence of $(a_n)_{n=1,2,\ldots}$ also converges to c.

Example 1. Let $a_n := (-1)^n$, $n \in \mathbb{N}$. We have $a_{2k} = 1$ and $a_{2k+1} = -1$ for all $k \in \mathbb{N}$. Thus, the subsequence $(a_{2k})_{k=1,2,\ldots}$ converges to 1 and the subsequence $(a_{2k+1})_{k=1,2,\ldots}$ converges to -1. Consequently, the sequence $(a_n)_{n=1,2,\ldots}$ diverges.

We are in a position to establish the following Bolzano-Weierstrass Theorem.

Theorem 4.1. *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let $(x_n)_{n=1,2,\ldots}$ be a bounded sequence of real numbers. We shall use mathematical induction to construct a nested sequence of closed intervals $(I_k)_{k=1,2,\ldots}$ as follows. Since $(x_n)_{n=1,2,\ldots}$ is bounded, $a_1 := \inf\{x_n : n \in \mathbb{N}\}$ and $b_1 := \sup\{x_n : n \in \mathbb{N}\}$ are real numbers. Let $I_1 := [a_1, b_1]$. Then $a_1 \le x_n \le b_1$ for all $n \in \mathbb{N}$. Let $E_1 := \{n \in \mathbb{N} : x_n \in I_1\}$. Then $E_1 = \mathbb{N}$ is an infinite set. Choose $c_1 := (a_1+b_1)/2$ to be the middle point of I_1. Then $I_1 = [a_1, c_1] \cup [c_1, b_1]$. If the set $\{n \in \mathbb{N} : x_n \in [a_1, c_1]\}$ is infinite, then let $a_2 := a_1$ and $b_2 := c_1$; otherwise, let $a_2 := c_1$ and $b_2 := b_1$. Let $I_2 := [a_2, b_2]$. Then in both cases, the set $E_2 := \{n \in \mathbb{N} : x_n \in I_2\}$ is infinite, for otherwise E_1 would be finite. Suppose that the intervals $I_1 = [a_1, b_1], I_2 = [a_2, b_2], \ldots, I_k = [a_k, b_k]$ have been constructed such that the set $E_k := \{n \in \mathbb{N} : x_n \in I_k\}$ is infinite. Choose $c_{k+1} := (a_k+b_k)/2$ to be the middle point of I_k. Then $I_k = [a_k, c_k] \cup [c_k, b_k]$. If the set $\{n \in \mathbb{N} : x_n \in [a_k, c_k]\}$ is infinite, then let $a_{k+1} := a_k$ and $b_{k+1} := c_k$; otherwise, let $a_{k+1} := c_k$ and $b_{k+1} := b_k$. Let $I_{k+1} := [a_{k+1}, b_{k+1}]$. Then in both cases, the set $E_{k+1} := \{n \in \mathbb{N} : x_n \in I_{k+1}\}$ is infinite. By our construction $I_{k+1} \subset I_k$ for every $k \in \mathbb{N}$ and $\lim_{k\to\infty}(b_k - a_k) = \lim_{k\to\infty}(b_1 - a_1)/2^{k-1} = 0$. By Theorem 3.2, there exists a real number c such that $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = c$. Let n_1 be the least element of the set E_1. Suppose that $n_1, \ldots, n_k$ have been chosen. Since the set E_{k+1} is infinite, the set $\{n \in E_{k+1} : n > n_k\}$ is also infinite. Let n_{k+1} be the least element of this set. Thus, we obtain an increasing sequence of positive integers $(n_k)_{k=1,2,\ldots}$. Let $y_k := x_{n_k}$ for $k \in \mathbb{N}$. We have $x_{n_k} \in I_k$, that is, $a_k \le x_{n_k} \le b_k$ for all $k \in \mathbb{N}$. Since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = c$, by Theorem 2.4 we conclude that $\lim_{k\to\infty} y_k = \lim_{k\to\infty} x_{n_k} = c$. This shows that $(x_n)_{n=1,2,\ldots}$ has a convergent subsequence. □

A sequence $(a_n)_{n=1,2,\ldots}$ of real numbers is called a **Cauchy sequence** if for each $\varepsilon > 0$ there exists a positive integer N such that

$$m, n > N \quad \text{implies} \quad |a_m - a_n| < \varepsilon.$$

Theorem 4.2. *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof. Suppose that $(x_n)_{n=1,2,\ldots}$ is a sequence of real numbers and $\lim_{n\to\infty} x_n = c$. For each $\varepsilon > 0$, there exists a positive integer N such that

$$n > N \quad \text{implies} \quad |x_n - c| < \varepsilon/2.$$

Consequently,

$$m, n > N \quad \text{implies} \quad |x_m - x_n| \le |x_m - c| + |c - x_n| < \varepsilon.$$

This shows that $(x_n)_{n=1,2,...}$ is a Cauchy sequence.

Now suppose that $(x_n)_{n=1,2,...}$ is a Cauchy sequence. We first prove that it is bounded. There exists a positive integer N such that $m, n > N$ implies $|x_m - x_n| < 1$. In particular, $|x_n - x_{N+1}| < 1$ for $n > N$, and so $|x_n| < |x_{N+1}| + 1$ for $n > N$. Let $M := \max\{|x_{N+1}| + 1, |x_1|, \dots, |x_N|\}$. Then $|x_n| \le M$ for all $n \in \mathbb{N}$.

Since the sequence $(x_n)_{n=1,2,...}$ is bounded. By Theorem 4.1, it has a subsequence $(x_{n_k})_{k=1,2...}$ that converges to some real number c. For each $\varepsilon > 0$, there exists a positive integer k_0 such that

$$k > k_0 \quad \text{implies} \quad |x_{n_k} - c| < \varepsilon/2.$$

Moreover, since $(x_n)_{n=1,2,...}$ is a Cauchy sequence, there exists a positive integer N such that

$$m, n > N \quad \text{implies} \quad |x_m - x_n| < \varepsilon/2.$$

Choose k such that $k > k_0$ and $n_k > N$. For $n > N$ we have

$$|x_n - c| \le |x_n - x_{n_k}| + |x_{n_k} - c| < \varepsilon.$$

This shows that $\lim_{n\to\infty} x_n = c$. □

Example 2. Let $a > 0$. If $(\alpha_n)_{n=1,2,...}$ is a Cauchy sequence of rational numbers, then the sequence $\left(a^{\alpha_n}\right)_{n=1,2,...}$ is convergent in $\mathbb{R}$.

Proof. By Theorem 4.2 it suffices to show that $\left(a^{\alpha_n}\right)_{n=1,2,...}$ is a Cauchy sequence. There are three possible cases: $a > 1$, $a = 1$, or $0 < a < 1$. If $a = 1$, then $a^{\alpha_n} = 1$ for all $n \in \mathbb{N}$, and the sequence $\left(a^{\alpha_n}\right)_{n=1,2,...}$ is convergent in $\mathbb{R}$. Let us consider the case $a > 1$. Since $(\alpha_n)_{n=1,2,...}$ is a Cauchy sequence, there exists a positive integer M such that $|\alpha_n| \le M$ for all $n \in \mathbb{N}$. It follows that $a^{-M} \le a^{\alpha_n} \le a^M$ for all $n \in \mathbb{N}$. Thus we have

$$\left|a^{\alpha_m} - a^{\alpha_n}\right| = \left|a^{\alpha_n}\left(a^{\alpha_m - \alpha_n} - 1\right)\right| \le a^M\left|a^{\alpha_m - \alpha_n} - 1\right|.$$

Let $\varepsilon > 0$ be given. Since $\lim_{k\to\infty} a^{1/k} = \lim_{k\to\infty} a^{-1/k} = 1$, there exists a positive integer K such that

$$1 - \varepsilon/a^M < a^{-1/K} < a^{1/K} < 1 + \varepsilon/a^M.$$

But $(\alpha_n)_{n=1,2,...}$ is a Cauchy sequence. So there exists a positive integer N such that $-1/K < \alpha_m - \alpha_n < 1/K$ whenever $m, n > N$. Suppose $m, n > N$. Then

$$1 - \varepsilon/a^M < a^{-1/K} < a^{\alpha_m - \alpha_n} < a^{1/K} < 1 + \varepsilon/a^M.$$

It follows that $\left|a^{\alpha_m-\alpha_n}-1\right| < \varepsilon/a^M$. Therefore,

$$m,n > N \Longrightarrow \left|a^{\alpha_m}-a^{\alpha_n}\right| \le a^M\left|a^{\alpha_m-\alpha_n}-1\right| < \varepsilon.$$

This shows that $\left(a^{\alpha_n}\right)_{n=1,2,\ldots}$ is a Cauchy sequence.

It remains to deal with the case $0 < a < 1$. In this case, we have $a^{\alpha_n} = (1/a)^{-\alpha_n}$ with $1/a > 1$. By what has been proved, the sequence $\left((1/a)^{-\alpha_n}\right)_{n=1,2,\ldots}$ is convergent in $\mathbb{R}$. Moreover, its limit is a positive real number, because $(1/a)^{-\alpha_n} \ge (1/a)^{-M} > 0$ for all $n \in \mathbb{N}$. Therefore, the sequence $\left(a^{\alpha_n}\right)_{n=1,2,\ldots}$ is convergent in $\mathbb{R}$. □

Now we can define the power a^α for any $a > 0$ and $\alpha \in \mathbb{R}$. Given $\alpha \in \mathbb{R}$, there exists a sequence $(\alpha_n)_{n=1,2,\ldots}$ of rational numbers such that $\lim_{n\to\infty} \alpha_n = \alpha$. We define

$$a^\alpha := \lim_{n\to\infty} a^{\alpha_n}.$$

If $(\beta_n)_{n=1,2,\ldots}$ is also a sequence of rational numbers such that $\lim_{n\to\infty} \beta_n = \alpha$. Then $\lim_{n\to\infty}(\beta_n - \alpha_n) = 0$. It follows that

$$\lim_{n\to\infty} a^{\beta_n} = \lim_{n\to\infty}\left(a^{\beta_n-\alpha_n}\cdot a^{\alpha_n}\right) = \left(\lim_{n\to\infty} a^{\beta_n-\alpha_n}\right)\cdot\left(\lim_{n\to\infty} a^{\alpha_n}\right) = \lim_{n\to\infty} a^{\alpha_n}.$$

Thus the power a^α is well defined. It is easily seen that the following properties hold for all $a, b > 0$ and $\alpha, \beta \in \mathbb{R}$:

$$a^\alpha \cdot a^\beta = a^{\alpha+\beta}, \quad \left(a^\alpha\right)^\beta = a^{\alpha\cdot\beta}, \quad (a\cdot b)^\alpha = a^\alpha \cdot b^\alpha.$$

A sequence $(x_n)_{n=1,2,\ldots}$ of real numbers is called **contractive** if there exists a real number q, $0 < q < 1$, such that

$$|x_{n+1} - x_n| \le q|x_n - x_{n-1}| \quad \forall\, n \ge 2.$$

Theorem 4.3. *Every contractive sequence of real numbers is a Cauchy sequence.*

Proof. Suppose that $(x_n)_{n=1,2,\ldots}$ is a contractive sequence such that the above inequality holds for some q with $0 < q < 1$. By mathematical induction we can show that

$$|x_{n+1} - x_n| \le q^{n-1}|x_2 - x_1| \quad \forall\, n \in \mathbb{N}.$$

For $m \ge 1$, by the triangle inequality we have

$$|x_{n+m} - x_n| = \left|\sum_{k=0}^{m-1}(x_{n+k+1} - x_{n+k})\right| \le \sum_{k=0}^{m-1}\left|x_{n+k+1} - x_{n+k}\right|.$$

It follows that

$$|x_{n+m} - x_n| \le \sum_{k=0}^{m-1} q^{n+k-1} |x_2 - x_1|.$$

Since $0 < q < 1$, we have

$$\sum_{k=0}^{m-1} q^{n+k-1} = q^{n-1} \sum_{k=0}^{m-1} q^k = q^{n-1}(1 + q + \cdots + q^{m-1}) = q^{n-1}\frac{1-q^m}{1-q} \le \frac{q^{n-1}}{1-q}.$$

Therefore,

$$|x_{n+m} - x_n| \le \frac{q^{n-1}}{1-q}|x_2 - x_1|.$$

But $\lim_{n\to\infty} q^{n-1} = 0$ because $0 < q < 1$, . This shows that $(x_n)_{n=1,2,\ldots}$ is a Cauchy sequence. □

By Theorem 4.2, $(x_n)_{n=1,2,\ldots}$ converges to a real number, say c. Fix n and let m go to ∞ in the inequality $|x_{n+m} - x_n| \le q^{n-1}/(1-q)|x_2 - x_1|$. Then we obtain the following estimate:

$$|c - x_n| \le \frac{q^{n-1}}{1-q}|x_2 - x_1| \quad \forall\, n \in \mathbb{N}.$$

Example 3. Let $(x_n)_{n=1,2,\ldots}$ be the sequence defined recursively as follows. Let $x_1 := 1$. For $n \ge 1$, let $x_{n+1} := 1/(2 + x_n)$. Then $(x_n)_{n=1,2,\ldots}$ is a contractive sequence.

Proof. First, we use mathematical induction to show that $x_n > 0$ for all $n \in \mathbb{N}$. If $n = 1$, then $x_1 = 1 > 0$. For the induction step, suppose $x_n > 0$. Then $2 + x_n > 0$; hence $x_{n+1} = 1/(2 + x_n) > 0$. This completes the induction procedure.

For $n \ge 2$ we have $x_{n+1} = 1/(2 + x_n)$ and $x_n = 1/(2 + x_{n-1})$. It follows that

$$x_{n+1} - x_n = \frac{1}{2 + x_n} - \frac{1}{2 + x_{n-1}} = \frac{(2 + x_{n-1}) - (2 + x_n)}{(2 + x_n)(2 + x_{n-1})} = \frac{x_{n-1} - x_n}{(2 + x_n)(2 + x_{n-1})}.$$

Since $x_n > 0$ for all $n \in \mathbb{N}$, we have $2 + x_n > 2$ for all $n \in \mathbb{N}$. Therefore,

$$|x_{n+1} - x_n| = \frac{|x_{n-1} - x_n|}{(2 + x_n)(2 + x_{n-1})} \le \frac{1}{4}|x_n - x_{n-1}|.$$

This shows that $(x_n)_{n=1,2,\ldots}$ is a contractive sequence.

By Theorem 4.3, $\lim_{n\to\infty} x_n = c$ for some $c \in \mathbb{R}$. Since $x_n > 0$ for all $n \in \mathbb{R}$, we have $c \ge 0$. Taking limits on both sides of the equation $x_{n+1} = 1/(2 + x_n)$, we obtain $c = 1/(2 + c)$. It follows that $c^2 + 2c - 1 = 0$. So $c = -1 + \sqrt{2}$ or $c = -1 - \sqrt{2}$. But $c \ge 0$. Therefore we must have $c = -1 + \sqrt{2}$. In other words, $\lim_{n\to\infty} x_n = \sqrt{2} - 1$. □

§5. Infinite Series

Given a sequence $(a_n)_{n=1,2,\ldots}$ of real numbers, define

$$s_n := \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n, \quad n \in \mathbb{N}.$$

We call s_n the nth **partial sum** of the **infinite series** $\sum_{n=1}^{\infty} a_n$.

If $(s_n)_{n=1,2,\ldots}$ converges to a real number s, we say that the series $\sum_{n=1}^{\infty} a_n$ converges and we write

$$\sum_{n=1}^{\infty} a_n = s.$$

The real number s is called the **sum** of the infinite series $\sum_{n=1}^{\infty} a_n$. If the sequence $(s_n)_{n=1,2,\ldots}$ diverges, then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges. If $\lim_{n\to\infty} s_n = \infty$, we say that the series $\sum_{n=1}^{\infty} a_n$ diverges to ∞ and write $\sum_{n=1}^{\infty} a_n = \infty$. If $\lim_{n\to\infty} s_n = -\infty$, we say that the series $\sum_{n=1}^{\infty} a_n$ diverges to $-\infty$ and write $\sum_{n=1}^{\infty} a_n = -\infty$.

If $1 \le m \le n$, then

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{m-1} a_k + \sum_{k=m}^{n} a_k.$$

Thus the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=m}^{\infty} a_n$ converges.

As an example, let us consider the series

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n} - \frac{1}{n+1} \Big).$$

Its nth partial sum is

$$s_n = \sum_{k=1}^{n} \Big(\frac{1}{k} - \frac{1}{k+1} \Big) = \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \cdots + \Big(\frac{1}{n} - \frac{1}{n+1}\Big) = 1 - \frac{1}{n+1}.$$

It follows that $\lim_{n\to\infty} s_n = 1$. Therefore, the series converges and its sum is 1.

The following results can be easily derived from the above definition.

Theorem 5.1. *If $\sum_{n=1}^{\infty} a_n = s$ and $\sum_{n=1}^{\infty} b_n = t$, then*

$$\sum_{n=1}^{\infty} (a_n + b_n) = s + t \quad \text{and} \quad \sum_{n=1}^{\infty} c a_n = cs \quad \text{for } c \in \mathbb{R}.$$

We observe that $a_n = s_n - s_{n-1}$ for $n \ge 2$. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $(s_n)_{n=1,2,\ldots}$ converges to a real number s. It follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = s - s = 0.$$

Thus, if a sequence $(a_n)_{n=1,2,...}$ diverges or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

If $a, r \in \mathbb{R}$ and $a_n = ar^{n-1}$ for $n \in \mathbb{N}$, then the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ar^{n-1}$$

is called a **geometric series**. The case $a = 0$ is trivial. In what follows we assume $a \neq 0$. If $|r| \geq 1$, then the sequence $(ar^{n-1})_{n=1,2,...}$ either diverges or converges to a nonzero real number. Hence, the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ diverges for $|r| \geq 1$. Suppose $|r| < 1$. Then

$$s_n = \sum_{k=1}^{n} ar^{k-1} = a(1 + r + \cdots + r^{n-1}) = \frac{a(1-r^n)}{1-r}, \quad n \in \mathbb{N}.$$

For $|r| < 1$ we have $\lim_{n\to\infty} r^n = 0$. Consequently,

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}.$$

Therefore, for $|r| < 1$, the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges and its sum is $a/(1-r)$.

We are in a position to consider infinite series with nonnegative terms.

Theorem 5.2. *Let $(a_n)_{n=1,2,...}$ be a sequence of real numbers with $a_n \geq 0$ for all $n \in \mathbb{N}$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence $(s_n)_{n=1,2,...}$ of partial sums is bounded.*

Proof. We have $s_n = a_1 + \cdots + a_n$. Since $a_n \geq 0$ for all $n \in \mathbb{N}$, $s_{n+1} \geq s_n$ for all $n \in \mathbb{N}$. Thus, $(s_n)_{n=1,2,...}$ is an increasing sequence. If this sequence is bounded, then it converges, by Theorem 3.1. Thus the series $\sum_{n=1}^{\infty} a_n$ converges if $(s_n)_{n=1,2,...}$ is bounded. If $(s_n)_{n=1,2,...}$ is unbounded, then the sequence diverges. Hence $\sum_{n=1}^{\infty} a_n$diverges. □

Let us investigate convergence or divergence of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is a real number. For $n \in \mathbb{N}$, let $a_n := 1/n^p$ and $s_n := a_1 + \cdots + a_n$. Suppose $p > 1$. The index set $\{j \in \mathbb{N} : 1 \leq j \leq 2^m - 1\}$ is the disjoint union $\cup_{k=1}^{m}\{j \in \mathbb{N} : 2^{k-1} \leq j \leq 2^k - 1\}$. It follows that

$$s_{2^m-1} = \sum_{k=1}^{m} \sum_{j=2^{k-1}}^{2^k-1} \frac{1}{j^p}.$$

If $2^{k-1} \leq j \leq 2^k - 1$, then $j^p \geq (2^{k-1})^p$ and $1/j^p \leq 1/(2^{k-1})^p$. The number of terms in the sum $\sum_{j=2^{k-1}}^{2^k-1} \frac{1}{j^p}$ is $2^k - 1 - 2^{k-1} + 1 = 2^{k-1}$. Hence,

$$\sum_{j=2^{k-1}}^{2^k-1} \frac{1}{j^p} \leq \frac{2^{k-1}}{(2^{k-1})^p} = \frac{2^{k-1}}{(2^p)^{k-1}} = (2/2^p)^{k-1} = (2^{1-p})^{k-1}.$$

Consequently,

$$s_{2^m-1} = \sum_{k=1}^{m} \sum_{j=2^{k-1}}^{2^k-1} \frac{1}{j^p} \le \sum_{k=1}^{m} (2^{1-p})^{k-1} < \frac{1}{1-2^{1-p}}.$$

Given $n \in \mathbb{N}$, we can find $m \in \mathbb{N}$ such that $n \le 2^m - 1$. So $s_n \le s_{2^m-1} < 1/(1-2^{1-p})$. This shows that the sequence $(s_n)_{n=1,2,\ldots}$ is bounded. By Theorem 5.2 the p-series converges for $p > 1$.

For $p \le 1$ and $m \in \mathbb{N}$ we have

$$s_{2^m} = \sum_{j=1}^{2^m} \frac{1}{j^p} \ge \sum_{j=1}^{2^m} \frac{1}{j} = 1 + \sum_{k=1}^{m} \sum_{j=2^{k-1}+1}^{2^k} \frac{1}{j} \ge 1 + \sum_{k=1}^{m} \frac{2^{k-1}}{2^k} = 1 + \frac{m}{2}.$$

Since $\lim_{m\to\infty}(1 + m/2) = \infty$, we see that the sequence $(s_n)_{n=1,2,\ldots}$ is unbounded. By Theorem 5.2 the p-series diverges for $p \le 1$.

§6. Convergence Tests for Series

In this section we give several tests for convergence of series.

Theorem 6.1 (Comparison Test). *Let $(a_n)_{n=1,2,\ldots}$ and $(b_n)_{n=1,2,\ldots}$ be two sequences of real numbers such that $0 \le a_n \le b_n$ for all $n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} b_n$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.*

Proof. For $n \in \mathbb{N}$, let $s_n := a_1 + \cdots + a_n$ and $t_n := b_1 + \cdots + b_n$. Since $0 \le a_n \le b_n$ for all $n \in \mathbb{N}$, we have $s_n \le t_n$ for all $n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} b_n$ converges, then the sequence $(t_n)_{n=1,2,\ldots}$ is bounded. Consequently, the sequence $(s_n)_{n=1,2,\ldots}$ is bounded. Therefore, the series $\sum_{n=1}^{\infty} a_n$ converges, by Theorem 5.2. □

Example 1. Test convergence or divergence for the series $\sum_{n=1}^{\infty} 2^{(-1)^n - n}$.
Solution. It follows from $(-1)^n \le 1$ that $2^{(-1)^n-n} \le 2^{1-n}$ for all $n \in \mathbb{N}$. Since the geometric series $\sum_{n=1}^{\infty} 2^{1-n}$ converges, the series $\sum_{n=1}^{\infty} 2^{(-1)^n-n}$ converges, by the comparison test.

Theorem 6.2 (The Ratio Test). *Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms such that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$$

exists. If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges. If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Suppose $L < 1$. Then there exists a real number q such that $L < q < 1$. Since $\lim_{n\to\infty} a_{n+1}/a_n = L$, there exists a positive integer N such that $a_{n+1}/a_n < q$ whenever

$n \geq N$. It follows that $a_{n+1} < qa_n$ for $n \geq N$. By using mathematical induction we infer that $a_n \leq q^{n-N}a_N$ for all $n \geq N$. Since $0 < q < 1$, the geometric series $\sum_{n=N}^{\infty} a_N q^{n-N}$ converges. By the comparison test, the series $\sum_{n=N}^{\infty} a_n$ converges. Therefore, the series $\sum_{n=1}^{\infty} a_n$ converges.

Suppose $L > 1$. Then there exists a real number r such that $1 < r < L$. Since $\lim_{n\to\infty} a_{n+1}/a_n = L$, there exists a positive integer N such that $a_{n+1}/a_n > r$ whenever $n \geq N$. It follows that $a_{n+1} > ra_n$ for $n \geq N$. By using mathematical induction we infer that $a_n \geq r^{n-N}a_N$ for all $n \geq N$. Since $r > 1$, the geometric series $\sum_{n=N}^{\infty} a_N r^{n-N}$ diverges. By the comparison test, the series $\sum_{n=1}^{\infty} a_n$ diverges. □

Example 2. Test convergence or divergence for the series $\sum_{n=1}^{\infty} n!/3^n$.
Solution. Let $a_n := n!/3^n$, $n \in \mathbb{N}$. We have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{3^{n+1}} \frac{3^n}{n!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{3^n}{3^{n+1}} = \lim_{n\to\infty} \frac{n+1}{3} = \infty.$$

By the ratio test, the series $\sum_{n=1}^{\infty} n!/3^n$ diverges.

Example 3. Let $0 < r < 1$. Test convergence or divergence for the series $\sum_{n=1}^{\infty} n^k r^n$, where k is a positive integer.
Solution. Let $u_n := n^k r^n$ for $n \in \mathbb{N}$. We have

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^k r^{n+1}}{n^k r^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^k r = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^k r = r.$$

Since $0 < r < 1$, the series $\sum_{n=1}^{\infty} n^k r^n$ converges, by the ratio test.

A series $\sum_{n=1}^{\infty} a_n$ is called an **alternating series** if there exists a sequence $(b_n)_{n=1,2,\ldots}$ of nonnegative numbers such that $a_n = (-1)^n b_n$ or $a_n = (-1)^{n-1} b_n$ for all $n \in \mathbb{N}$.

Theorem 6.3 (The Alternating Series Test). *If $(b_n)_{n=1,2,\ldots}$ is a sequence of nonnegative numbers such that $b_n \geq b_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = 0$, then the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.*

Proof. Let $s_n := \sum_{k=1}^{n} (-1)^{k-1} b_k$, $n \in \mathbb{N}$. We claim that $(s_{2n})_{n=1,2,\ldots}$ is an increasing sequence. Indeed, we have

$$s_{2n+2} - s_{2n} = (-1)^{2n} b_{2n+1} + (-1)^{2n+1} b_{2n+2} = b_{2n+1} - b_{2n+2} \geq 0.$$

Moreover, $(s_{2n+1})_{n=1,2,\ldots}$ is a decreasing sequence, because

$$s_{2n+3} - s_{2n+1} = (-1)^{2n+1} b_{2n+2} + (-1)^{2n+2} b_{2n+3} = -b_{2n+2} + b_{2n+3} \leq 0.$$

Further, $s_{2n+1} - s_{2n} = (-1)^{2n} b_{2n+1} = b_{2n+1} \geq 0$, so $s_{2n} \leq s_{2n+1}$ for all $n \in \mathbb{N}$. By Theorem 3.1, both sequences $(s_{2n})_{n=1,2,\ldots}$ and $(s_{2n+1})_{n=1,2,\ldots}$ converge. But

$$\lim_{n\to\infty} (s_{2n+1} - s_{2n}) = \lim_{n\to\infty} b_{2n+1} = 0.$$

Thus, there exists a real number s such that $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n+1} = s$. Consequently, $\lim_{n\to\infty} s_n = s$. This shows that the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges. □

Note that $s_{2n} \leq s \leq s_{2n+1}$ for all $n \in \mathbb{N}$. It follows that

$$0 \leq s - s_{2n} \leq s_{2n+1} - s_{2n} = b_{2n+1} \quad \text{and} \quad 0 \leq s_{2n+1} - s \leq s_{2n+1} - s_{2n+2} = b_{2n+2}.$$

Thus we get the following error estimate:

$$|s - s_n| \leq b_{n+1} \quad \forall\, n \in \mathbb{N}.$$

Example 4. For $p > 0$, the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1}/n^p$ converges.
Proof. For $n \in \mathbb{N}$, let $b_n := 1/n^p$ and $a_n := (-1)^{n-1} b_n$. The sequence $(b_n)_{n=1,2,\ldots}$ is decreasing. Indeed, since $p > 0$, we have $n^p \leq (n+1)^p$, so $1/n^p \geq 1/(n+1)^p$ for all $n \in \mathbb{N}$. Moreover, $\lim_{n\to\infty} 1/n^p = 0$. By the alternating series test, the series $\sum_{n=1}^{\infty} (-1)^{n-1}/n^p$ converges.

For a real number a, let $a^+ := \max\{a, 0\}$ and $a^- := \max\{-a, 0\}$. We call a^+ the **positive part** of a and a^- the **negative part** of a, respectively. Evidently, $|a| = a^+ + a^-$ and $a = a^+ - a^-$.

Theorem 6.4. *Let $(a_n)_{n=1,2,\ldots}$ be a sequence of real numbers. If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.*

Proof. We observe that $0 \leq a_n^+ \leq |a_n|$ and $0 \leq a_n^- \leq |a_n|$ for all $n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then both $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ converge, by the comparison test. But $a_n = a_n^+ - a_n^-$ for all $n \in \mathbb{N}$. We conclude that the series $\sum_{n=1}^{\infty} a_n$ converges. □

If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the series $\sum_{n=1}^{\infty} a_n$ **converges absolutely**. If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges, then we say that $\sum_{n=1}^{\infty} a_n$ **converges conditionally**. For example, the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges conditionally.